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Noise Sensitivity And Exceptional Times

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Noise Sensitivity And Exceptional Times

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Summary

In this thesis we study the behavior of some stochastic processes that evolve over time via rerandomisations at times corresponding to independent exponential clocks. Comparing a given process at any pair of times (s, t) is equivalent to "noising" the process by a parameter $\varepsilon := \varepsilon_{|s-t|}$.

The main pair of objects we consider are two one-dimensional simple symmetric random walks denoted (Y_n) and (Z_n). The former moves up or down with probability $1/2$ each, while the latter continues in the direction it is already moving in with probability $1/2$ and turns around otherwise. Despite both having the same law as sequences, they display very different behavior when they evolve over time, or are noised. We also study a continuous space Brownian motion version of (Z_n), and study the time evolution of that process.

A branching random walk is also considered, where each particle has exactly two offspring and moves as a random walk that either goes right one position or stays still. We allow this process to evolve over time and study how quickly the left-most particle diverges to infinity.

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My office mates Bas Lodewijks, Isaac Gonzáles and Andrea Lelli get a special mention, as their wits and humour undoubtedly kept my sanity in check these past few years.

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Chapter 1

Introduction

1.1 Noise sensitivity

Noise sensitivity was first studied in the probability literature in 1999, in a paper by Benjamini, Kalai, and Schramm [8]. This makes it a relatively new and exciting field within modern probability theory, and it has distinct overlaps with statistical physics and computer science. An excellent reference for an overview of noise sensitivity theory is [19] by Garban and Steif.

1.1.1 Definitions and examples

Consider the n-dimensional hypercube $\Omega_n := \{-1,1\}^n$, where we say an element $\omega = (x_1,\ldots,x_n) \in \Omega_n$ is a sequence of n bits. A Boolean function is a map $f : \Omega_n \to \{0,1\}$. In the area of voting models, where Boolean functions are commonly used, each bit can be thought of as an individual voting for candidate A or B, represented by 1 and -1 respectively. Then the Boolean function f can be viewed as the voting system that collates said votes and outputs the winner of said vote. We shall see a few examples that can be viewed in this way shortly.

It can also be helpful to think of f as an indicator function on n bits. To be exact, $f = \mathbb{1}_{A_f}$ where

$$A_f = \{\omega \in \{-1,1\}^n \ : \ f(\omega) = 1\}.$$

We now define noise sensitivity and noise stability of a *sequence* of boolean functions (f_n), where $f_n : \{-1,1\}^{m_n} \to \{0,1\}$ for an increasing sequence of positive integers (m_n). Let X be chosen uniformly from $\{-1,1\}^{m_n}$, equivalently denote $X = (X_1,\ldots,X_{m_n})$

where the X_i are IID Rademacher random variables. That is,

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2 \quad \forall i \in \mathbb{N}.$$

Again, we call the X_i "bits". Fix $\varepsilon \in (0,1)$, let X_ε be X but with each X_i independently rerandomised with probability ε. Equivalently, we can say that each X_i becomes $-X_i$ with probability $\varepsilon/2$. We view the process of moving from X to X_ε as "noising".

We say that (f_n) is called noise sensitive if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} (\mathbb{E}[f_n(X)f_n(X_\varepsilon)] - \mathbb{E}[f_n(X)]^2) = 0.$$

If the above holds for a sequence (ε_n) where $\varepsilon_n \to 0$, rather than just a fixed ε, we say that (f_n) is quantitatively noise sensitive with respect to (ε_n). Quantitative noise sensitivity is evidently stronger than noise sensitivity.

Viewing $f_n = \mathbb{1}_{A_{f_n}}$, (f_n) being noise sensitive corresponds to (A_{f_n}) being asymptotically independent in the sense that for large n, knowing whether or not A_{f_n} occurs pre-rerandomisation would not help you determine whether it occurs after rerandomisation.

The sequence (f_n) is noise stable if

$$\lim_{\varepsilon\to 0} \sup_n \mathbb{P}(f_n(X) \neq f_n(X_\varepsilon)) = 0.$$

(f_n) being noise stable corresponds to (uniformly in n) the events (A_{f_n}) being almost identical as $\varepsilon \to 0$.

Note that noise sensitivity and noise stability are not opposites, there exist Boolean functions that are both noise sensitive and noise stable, and functions that are neither. The former occurs iff $Var(f_n) \to 0$ as $n \to \infty$, i.e. $f_n \to C$, a constant, in probability. We now discuss a few important examples.

Example 1 (Dictator)**.** Define $\mathbf{DICT}_n : \Omega_n \to \{0,1\}$ by

$$\mathbf{DICT}_n(x_1, \ldots, x_n) = \mathbb{1}_{\{x_1=1\}}.$$

In other words, the first bit is a dictator as it determines the value of the function, overruling all other bits. Writing the ith bit of X_ε as $X_{\varepsilon,i}$, the dictator function is noise stable as $\mathbb{P}(\mathbf{DICT}_n(X) \neq \mathbf{DICT}_n(X_\varepsilon)) = \mathbb{P}(X_1 \neq X_{\varepsilon,1}) = \varepsilon/2$ which is independent of n and tends to zero as $\varepsilon \to 0$.

Example 2 (Parity)**.** Define $\mathbf{PAR}_n : \Omega_n \to \{0,1\}$ by

$$\mathbf{PAR}_n(x_1, \ldots, x_n) = \text{sign}\left(\prod_{i=1}^{n} x_i\right)$$

where $sign(x) = 1$ if $x \geq 0$ and 0 otherwise. So $\mathbf{PAR}_n$ takes the value 1 if there are an even number of -1-valued bits, and 0 otherwise. We prove the parity function is noise sensitive and not noise stable. We use the standard fact that the probability generating function of a $B = Bin(n,p)$ random variable is $G(z) = ((1-p)+pz)^n$ and so $\mathbb{P}(B \text{ is even}) = (1/2)(G(1)+G(-1)) = (1/2)(1+(1-2p)^n)$. We now compute the three quantities:

$$\mathbb{E}[\mathbf{PAR}_n(\omega)] = \mathbb{P}(\mathbf{PAR}_n(\omega) = 1) = \mathbb{P}(Bin(n, 1/2) \text{ is even}) = \frac{1}{2}$$

and

$$\mathbb{P}(\mathbf{PAR}_n(\omega) \neq \mathbf{PAR}_n(\omega_\varepsilon)) = \mathbb{P}(Bin(n, \varepsilon/2) \text{ is odd}) = \frac{1}{2}(1-(1-\varepsilon)^n)$$

and

$$\begin{aligned}\mathbb{E}[\mathbf{PAR}_n(\omega)\mathbf{PAR}_n(\omega_\varepsilon)] &= \mathbb{P}(\mathbf{PAR}_n(\omega) = \mathbf{PAR}_n(\omega_\varepsilon) = 1) \\ &= \mathbb{P}(\mathbf{PAR}_n(\omega) = 1)\mathbb{P}(\mathbf{PAR}_n(\omega_\varepsilon) = 1 \mid \mathbf{PAR}_n(\omega) = 1) \\ &= \frac{1}{2}\mathbb{P}(Bin(n, \varepsilon/2) \text{ is even}) = \frac{1}{4}(1+(1-\varepsilon)^n)\end{aligned}$$

where we've used that $\mathbf{PAR}_n$ changes sign after the noise iff an odd number of bits change due to the noise, which again, can be viewed as a binomial random variable. Plugging these quantities into the two definitions (and ensuring you take the supremum over n *first* in the noise stable definition) reveals that the Parity function is noise sensitive and not noise stable.

Example 3 (Majority)**.** Define $\mathbf{MAJ}_n : \Omega_n \to \{0,1\}$ by

$$\mathbf{MAJ}_n(x_1, \ldots, x_n) = \text{sign}\left(\sum_{i=1}^{n} x_i\right)$$

where $\text{sign}(x) = 1$ if $x \geq 0$ and 0 otherwise. This example corresponds to a democratic system where every individual's vote is weighted equally. This function is not noise sensitive which we shall prove in the next subsection, while it is noise stable which we prove now. The proof we use is similar to the proof given in [8, Theorem 1.8], which is of a far more general result, but we do use ideas that come up in proofs in the next two chapters.

Given a random Rademacher vector $X = (X_1, \ldots, X_n)$, write $Y_n = \sum_{i=1}^{n} X_i$ (an object we call the compass random walk, to be introduced later), and denote the sum of the noised Rademachers by $Y_n(\varepsilon)$. Let J denote the random subset of $[n] = \{1, \ldots, n\}$ consisting of the indices of bits that have changed between Y_n and $Y_n(\varepsilon)$. Writing

$$U_n = \sum_{i \notin J} X_i \qquad V_n = \sum_{i \in J} X_i$$

we have that $Y_n = U_n + V_n$ and $Y_n(\varepsilon) = U_n - V_n$. It can be seen that if the signs of Y_n and $Y_n(\varepsilon)$ vary then we must have that $|Y_n| < 2|V_n|$. Therefore it suffices to show that $\lim_{\varepsilon \to 0} \sup_n \mathbb{P}(|Y_n| < 2|V_n|) = 0$. If the supremum occurred at some finite $N \in \mathbb{N}$, then as $\varepsilon \to 0$, $V_N \to 0$ in probability so our probability would be zero as required. Turning solely to the case where the supremum occurs as $n \to \infty$, we split based on the size of V_n to obtain

$$\mathbb{P}(|Y_n| < 2|V_n|) \leq \mathbb{P}(|Y_n| < 2\varepsilon^{1/4}\sqrt{n}) + \mathbb{P}(|V_n| \geq \varepsilon^{1/4}\sqrt{n}).$$

For the first term, we use the Central Limit Theorem which gives (ignoring minor technicalities regarding ceiling functions)

$$\mathbb{P}(|Y_n| < 2\varepsilon^{1/4}\sqrt{n}) = 2\mathbb{P}\left(0 \leq \frac{1}{\sqrt{n}} Y_n < 2\varepsilon^{1/4}\right) \to \frac{2}{\sqrt{2\pi}} \int_0^{2\varepsilon^{1/4}} e^{-\frac{x^2}{2}} \, \mathrm{d}x \leq \frac{4\varepsilon^{1/4}}{\sqrt{2\pi}}.$$

For the second term, we start by conditioning on $|J|$ and use Markov's inequality:

$$\mathbb{P}(|V_n| \geq \varepsilon^{1/4}\sqrt{n} \mid |J|) = \mathbb{P}(V_n^2 \geq \sqrt{\varepsilon} n \mid |J|) \leq \frac{1}{\sqrt{\varepsilon} n} \mathbb{E}[V_n^2 \mid |J|] = \frac{1}{\sqrt{\varepsilon} n} |J|.$$

Noting that $|J| \sim Bin(n, \varepsilon/2)$, taking expectations of the above gives

$$\mathbb{P}(|V_n| \geq \varepsilon^{1/4}\sqrt{n}) \leq \frac{1}{\sqrt{\varepsilon} n} \mathbb{E}[|J|] = \frac{1}{2}\sqrt{\varepsilon}.$$

Both terms are independent of n and tend to zero, which suffices.

1.1.2 Influence and BKS

Given a Boolean function $f : \Omega_n \to \{0, 1\}$ and $m \in [n] = \{1, \ldots, n\}$, we say that the mth bit is pivotal for f for $\omega \in \Omega_n$ if $f(\omega) \neq f(\omega^m)$ where ω^m is ω with the mth bit changed. Now the influence of the mth bit is defined as

$$\mathcal{I}_m(f) := \mathbb{P}(f(X) \neq f(X^m))$$

the probability that m is pivotal for f (where ω is chosen uniformly at random). The total influence is

$$\mathcal{I}(f) = \sum_{m=1}^{n} \mathcal{I}_m(f).$$

The BKS Theorem discovered by Benjamini, Kalai and Schramm [8] is a powerful result that shows that a class of Boolean functions is immediately noise sensitive:

Theorem 1.1. *A sequence of Boolean functions (f_n) is noise sensitive if*

$$\lim_{n\to\infty} \sum_m \mathcal{I}_m(f_n)^2 = 0.$$

This result is an "iff" result provided that the boolean sequence (f_n) consists of monotonic functions. That is, if $\omega_1 \leq \omega_2$ then $f_n(\omega_1) \leq f_n(\omega_2)$ for all n, where $\omega_1 \leq \omega_2$ means that every bit in ω_1 is $\leq$ every corresponding bit in ω_2. $\leq$ defines a partial ordering of elements of Ω_n.

We shall not use the BKS theorem throughout the bulk of this thesis, but it shall be used in the next section to show noise sensitivity of percolation crossing events. However, the idea of influences is an important one that we shall use when producing upper bounds on our Hausdorff dimensions in Chapters 2 and 3.

1.1.3 Dynamical processes and exceptional times

We now introduce the second way we can alter a stochastic process, by allowing it to evolve over time via specific dynamics. These dynamics were used by Häggström, Peres and Steif in [36] to study dynamical percolation, which we shall give an overview of in the next section.

Informally, we set up our dynamics by attaching a sequence of exponential clocks of rate 1 onto each bit, and whenever a clock rings, the bit corresponding to that clock rerandomises its value. To be precise, for each $j \geq 1$, let $(N_j(t), t \geq 0)$ be an independent Poisson process of rate 1, and for each $i \geq 0$, let X_j^i be an independent random variable with $\mathbb{P}(X_j^i = 1) = \mathbb{P}(X_j^i = -1) = 1/2$. Then define

$$X_j(t) = X_j^i \quad \text{whenever} \quad N_j(t) = i.$$

In words, $X_j(t)$ has the same distribution as X_j and rerandomises itself at the times of the Poisson process $N_j(t)$. We can then define $X(t) = (X_1(t), X_2(t), \dots)$ to be our dynamical random variable of bits rerandomising over time.

There is a direct relationship between this model and "noising" that we introduced earlier. If we have $X(s)$ and $X(s+t)$ for some $t > 0$ then by the Markov property and basic facts about the exponential distribution we have that the joint distribution of these two random variables is that same as that of X and X_ε for $\varepsilon = 1 - e^{-t}$. Equally, given $\varepsilon \in (0,1)$ we can define $t = -\log(1-\varepsilon)$. We generally consider small values of t, or equivalently, small values of ε so the approximation $1 - e^{-t} \approx t$ is usually appropriate for intuition's sake. In particular, we may compare $f_n(X(0))$ with $f_n(X(t))$ (written $f_n(0)$ and $f_n(t)$ in short) when deciding whether (f_n) is noise sensitive or noise stable.

Let $F = (F_n)$ be a stochastic process where $F_n = F_n(X_1, \dots, X_{m_n})$ is built using $X_1, \dots X_{m_n}$. Recall that (m_n) is a positive increasing sequence. Let A be an event that F satisfies almost surely, that is, $\mathbb{P}(F \text{ satisfies } A) = 1$. Writing $F(t) = (F_n(X_1(t), \dots, X_{m_n}(t)))_n$ via the above dynamics, as $F(t) = F$ in law we have

$$\forall t \ \mathbb{P}(F(t) \text{ satisfies } A) = 1.$$

It can also be shown by Fubini's Theorem that

$$\mathbb{P}(F(t) \text{ satisfies } A \text{ for } a.e. \ t) = 1.$$

The fundamental question of interest then becomes where or not we can replace "*a.e.* t" with "$\forall t$"? If so we say that A is dynamically stable with respect to $(F(t))_t$, otherwise A is dynamically sensitive and there are exceptional times where A^c occurs with positive probability. If certain ergodic (or zero-one law) arguments can be applied then this can be strengthened to there being exceptional times almost surely. Notice that the set of exceptional times has Lebesgue measure zero, meaning that to get a grasp of the "size" of the set of exceptional times we need some other measure. For this purpose we use the Hausdorff dimension. Fix $E \subset \mathbb{R}$. For $d \geq 0$, $\delta > 0$ define

$$\mathcal{H}^d_\delta(E) := \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } C_i)^d \ : \ E \subset \bigcup_{i=1}^{\infty} C_i \text{ and diam } C_i < \delta \ \forall i \right\}$$

where diam C_i is the maximum (Euclidean) distance between any two points in C_i, and the infimum is taken over all valid open covers of E (a single cover is given by (C_i)).

Now, the Hausdorff dimension of E is given by

$$\dim_H(E) := \inf\{d \geq 0 \;:\; \mathcal{H}^d(E) = 0\}$$

where

$$\mathcal{H}^d(E) := \lim_{\delta \to 0} \mathcal{H}^d_\delta(E).$$

We call $\mathcal{H}^d_\delta(E)$ the Hausdorff content of E. For general theory regarding the Hausdorff dimension, the interested reader is steered towards the following book by Falconer [17]. We rarely use the above definition throughout this thesis. Instead, the main result we use is that the Hausdorff dimension of a countable union is equal to the supremum of the individual Hausdorff dimensions. See the discussion before [17, Proposition 2.3] for a proof.

Dynamically sensitive events are an incredibly interesting object of study, as they are events that a.s. do not occur and any fixed time, but almost surely occur at some random time. There is an informal link between noise sensitivity and dynamical sensitivity in that dynamically sensitive events seem to have a corresponding noise sensitive function, but there have been no rigorous result proven regarding this relationship.

1.2 Dynamical percolation and crossings

This section, which is disjoint from the rest of this thesis, covers the percolation model that put the theory of noise sensitivity and dynamical sensitivity into the spotlight. Without the following work, this thesis would likely have not been produced at all. These dynamics were used by Häggström, Peres and Steif in [36].

For a general overview of percolation theory, we recommend the following book from Grimmett [21]. For a general guide regarding percolation on the graphs we are considering, as well as the noise sensitivity and exceptional times results, see [19] by Garban and Steif. [48] also provides a good reference to material on dynamical percolation.

1.2.1 Percolation on $\mathbb{T}$ and $\mathbb{Z}^2$

$\mathbb{T}$ is defined by the set of points

$$\mathbb{Z} + e^{i\pi/3}\mathbb{Z} := \{x + e^{i\pi/3}y \;:\; x, y \in \mathbb{Z}\},$$

thus we are embedding $\mathbb{T}$ in $\mathbb{C}$. We now define site percolation on $\mathbb{T}$. The site of a vertex $x \in \mathbb{T}$ consists of all points in $\mathbb{C}$ that are closer to x in Euclidean distance than any other vertex in $\mathbb{Z} + e^{i\pi/3}\mathbb{Z}$. These sites are hexagonal in shape. For site percolation, fix $p \in [0,1]$ and with probability p keep each site, independently of each other, i.e. discard each site with probability $1-p$. The resulting graph is a realisation of percolation on $\mathbb{T}$. The sites kept are called "open" and the edges thrown out are "closed".

We also define bond percolation on $\mathbb{Z}^2$ as follows. The graph structure consists of the elements of $\mathbb{Z}^2$ as vertices, with edges connecting every pair of vertices that are at a Euclidean distance of one from each other. The bond percolation model on $\mathbb{Z}^2$ is as follows, fix $p \in [0,1]$, then keep each edge in $\mathbb{Z}^2$ independently with probability p, and discard them otherwise. The resulting graph is a particular realisation of percolation on $\mathbb{Z}^2$.

Throughout this section we focus solely on $\mathbb{T}$, however the results to come can be somewhat applied to bond percolation on $\mathbb{Z}^2$. These results tend to be less accurate as site percolation on $\mathbb{T}$ is known to be conformally invariant in the scaling limit while bond percolation on $\mathbb{Z}^2$ is believed to be, but this has not yet been proven. We will mention any significant differences between the results for both graphs when appropriate.

The main question when considering percolation models is whether or not an infinite connected component of edges/sites exists. Given a particular graph structure G and the origin 0, we denote by $C(0)$ the open component of edges/sites containing 0, then we can define

$$\theta_G(p) := \mathbb{P}_p(|C(0)| = \infty)$$

where $\mathbb{P}_p$ denotes the probability measure for our percolation model with parameter p. We can also define the percolation threshold for a graph G

$$p_c(G) := \sup\{p \ : \ \theta_G(p) = 0\}.$$

It is known that if $p < p_c(G)$ then there almost surely does not exist an infinite open component in G, while if $p > p_c(G)$ then there almost surely does exist such a component. It can be shown that a $0-1$ law applies when $p = p_c(G)$ so that infinite components either exist almost surely or not.

We have that for bond percolation $p_c(\mathbb{Z}^2) = 1/2$ and that for site percolation $p_c(\mathbb{T}) = 1/2$. It is also known that when $p = p_c$ in either case we almost surely do not have infinite components, see [22, Theorem 1] and [24, Theorem 3.1 and (3.67)] respectively.

As this system is discrete (there are a countable number of sites), we can "noise" it in a way that allows us to ask questions regarding noise sensitivity. To be explicit, after a given realisation of the percolation model with parameter p, we fix $\varepsilon > 0$ and rerandomise every site (both open and closed) with probability ε. That is, if rerandomised it becomes open with probability p and closed otherwise.

This also means that we can study percolation on a graph that evolves over time as in Section 1.1.3. To reiterate, we can attach every site with independent Bernoulli(p) random variables (a value of 1 corresponding to being on), and then have each of these site variables rerandomise at independent $Exp(1)$ times.

From here on out, we fix $p = p_c = 1/2$. The reason for this is that by [36, Proposition 1.1], when we are away from criticality there are almost surely no exceptional times for percolation events.

We now define the Boolean function of interest this section, the left-right crossing function. Fix $a, b > 0$ and consider, for each $n \in \mathbb{N}$, the rectangle $R_n := [0, an] \times [0, bn]$. We define the Boolean function $f_n : \{-1, 1\}^{O(1)n^2} \to \{0, 1\}$ on $\mathbb{T}$ (resp. $\mathbb{Z}^2$) as the indicator function of whether or not a left-to-right crossing of open sites (resp. edges) in the rectangle R_n, where this rectangle is intersected with the respective graph's structure. See Figure 1-1 for an example.

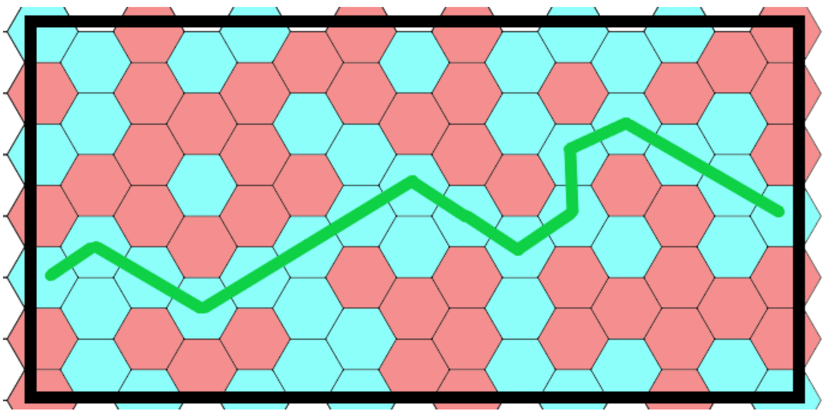

Figure 1-1: A left-to-right crossing on $\mathbb{T}$ within a given rectangle. Sites that are open are blue, while red sites are closed.

The main result here is a combination of [19, Corollary 8.3, Theorem 8.4] and is stated as follows:

Theorem 1.2. (f_n) *defined as above is noise sensitive for both* $\mathbb{T}$ *and* $\mathbb{Z}^2$*.*

Recall that we can write $f_n = \mathbb{1}_{A_{f_n}}$ where A_{f_n} is the event that there is a left-right crossing on R_n. As we have seen, a Boolean function is trivially both noise sensitive and noise stable if $\mathbb{P}(A_{f_n}) \to 0$ or 1. The RSW theorem [19, Theorem 2.1] says that for fixed a, b, there exists a constant $c = c(a, b) > 0$ such that

$$c < \mathbb{P}(A_{f_n}) < 1 - c \quad \forall n \in \mathbb{N}.$$

Not only does this tell us that asking about noise sensitivity is non-trivial, but it also means that there is a non-zero probability of seeing open clusters of any finite size. We also have the following result, see [19, Theorem 11.6].

Theorem 1.3. *There almost surely exist exceptional times where $\mathbb{T}$ has an open infinite component, and the set of such exceptional times has Hausdorff dimension* $31/36$*.*

The above also holds for $\mathbb{Z}^2$ except the Hausdorff dimension is unknown. A first moment argument not detailed here provides an upper bound on the Hausdorff dimension of such times of $31/36$, but no lower bound.

In the following subsections, we briefly detail a few different methods of trying to prove Theorems 1.2 and 1.3. None of this machinery is used elsewhere in the thesis.

1.2.2 Method 1: BKS

Here we utilise the BKS Theorem, Theorem 1.1, to prove Theorem 1.2. Sadly, the BKS theorem does not help us prove the existence of exceptional times as in Theorem 1.3. See Chapter 6 of [19] for a more detailed version of the following computations.

In order to utilise the BKS Theorem we need to have a solid understanding of the influences of (f_n), i.e.

$$\mathcal{I}_m(f_n) = \mathbb{P}(m \text{ is pivotal for } f_n)$$

where m is a site in $\mathbb{T}$. Recall than m being pivotal for f_n means that whether m is on or off changes whether or not there is a left-to-right crossing on $R_n = [0, an] \times [0, bn]$

For simplicity's sake, take m to be a site far from the boundary ∂R_n. For m to be pivotal, we need an open path from m to the left boundary and an open path from m to the right boundary, so that when m is on we definitely have a crossing. However, we also need a closed path from m to the top and bottom boundary, so that when m is off, R_n is split down the middle so that a left-right crossing cannot exist.

Therefore, at a bare minimum, we need four paths of alternating "colours" (say, white is on and black is off) from m in $B_R(m)$, the ball of radius R with centre m, where R is the distance from m to ∂R_n. This event we call a four-arm event, and the probability it occurs is denoted $\alpha_4(R)$. It can be see in Werner's works [46, 52] (the former done with Smirnov), that for all $\varepsilon > 0$, $\exists C > 0$ such that for all $R \geq 1$ and $r > 0$

$$\alpha_4(R) \leq CR^{-5/4+\varepsilon}$$

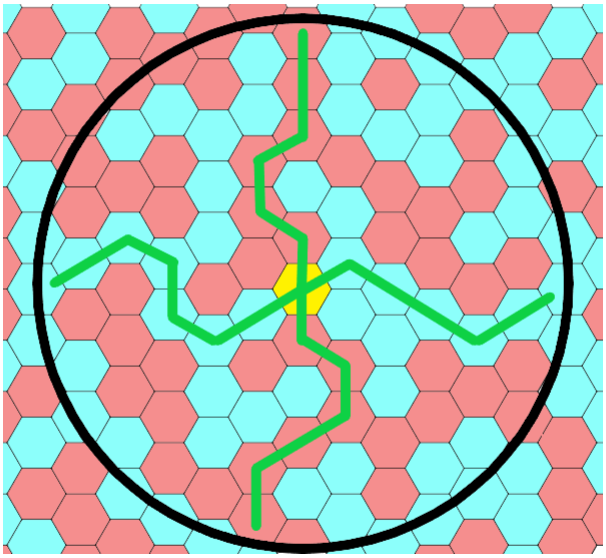

Figure 1-2: A four-arm event on a circle of radius R which occurs with probability $\alpha_4(R)$. Blue corresponds to on and red corresponds to off. Notice that the status of the yellow site determines where or not there is a blue left-to-right crossing and a red top-to-bottom crossing.

The four-arm event is a fairly good approximation for the influence for most sites m, this is because most sites are not "too close" to the boundary of R_n. In the case that m is close to ∂R_n, but still somewhat away from a corner, it is better to consider the three-arm event in the half-plane. That is, the probability we have three paths of alternating colours from the boundary of a half-plane to radius R out. The case that m is close to a corner is best approximated by a two-arm event in the quarter plane, defined similarly. Bounding these other events and then combining them via a technical argument leads to the following bound on the sum of the squared influences:

$$\sum_{m \in R_n} \mathcal{I}_m(f_n)^2 \leq Cn^2(n^{-5/4+\varepsilon})^2 = Cn^{-1/2+2\varepsilon}$$

and we are free to choose any $\varepsilon > 0$. So choosing $\varepsilon < 1/4$ gives us the convergence to zero that BKS desires.

1.2.3 Method 2: Randomised algorithms

This second method, randomised algorithms, utilises the connection Boolean functions have to computer science and allow us to learn about both noise sensitivity and exceptional times. See Chapters 8 and 11 of [19] for a more detailed guide to the theory and algorithms stated.

Given a Boolean function $f : \Omega_n \to \{0, 1\}$, and a randomly chosen n-string $\omega \in \Omega_n$, an algorithm (or decision tree) on f is a function A of both f and the revealed bits of ω, which reveals bits of ω in a particular order (dependent on both A and the position/values of previously revealed bits were) and stops when the value of $f(\omega)$ is known. A randomised algorithm (or randomised decision tree) is exactly the same, but external randomness may also be used when deciding which bits to reveal.

Let J_A be the random set of bits revealed by the randomised algorithm A, and define the revealment of A as

$$\delta_A := \max_{i \in [n]} \mathbb{P}(i \in J_A)$$

i.e. the probability that the bit which is most likely to be revealed, is revealed. The revealment of the function f is

$$\delta_f := \inf_{A} \delta_A$$

the smallest revealment over all possible randomised algorithms of f. [19, Corollary 8.3] states that a sequence of Boolean functions (f_n) is noise sensitive if

$$\lim_{n \to \infty} \delta_{f_n} = 0.$$

In other words, noise sensitivity occurs when (in the limit) no particular bit of ω has a sizable influence over the value of f_n. The randomised algorithm used to show noise sensitivity is as follows:

1. Pick a random site from the middle third of the right hand boundary of the rectangle under consideration. This step needs auxiliary randomness so that the revealment of the starting site is not too large.

2. From the right of that hexagon, create a path moving upwards along the edges of the hexagons that turns left at closed hexagons and right at open ones, stopping when we reach the left-hand side or get stuck at another boundary. Note that we are querying the hexagons required to make such a path.

3. From that same initial position, create a downwards path that turns right at

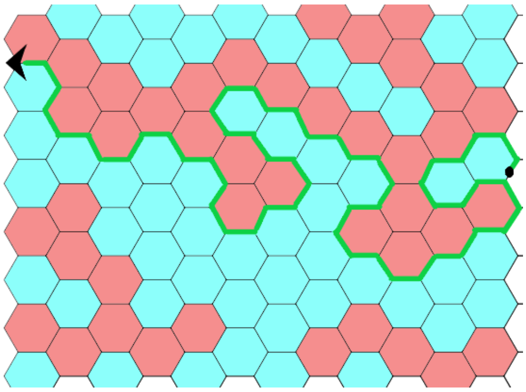

Figure 1-3: A realisation of the (green) path along the edges traced out by step 2 of the algorithm.

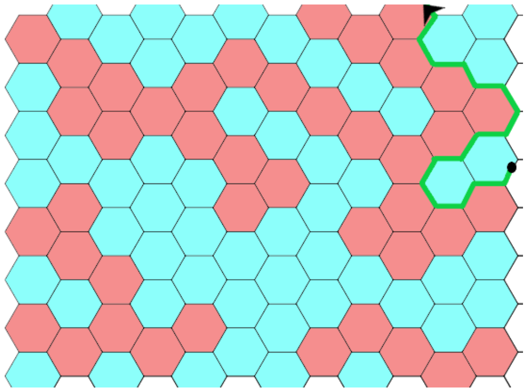

Figure 1-4: A realisation of the (green) path along the edges traced out by step 3 of the algorithm.

closed hexagons and left at open ones. We again stop when we get stuck or reach the left-hand side.

4. If either path succeeds in reaching the left-hand side then we know $f_n(\omega) = 1$, otherwise we have $f_n(\omega) = 0$.

See figures 1-3 and 1-4 for a demonstration of this algorithm. As the paths traced out by this algorithm run on the boundary of open/closed components, we know that if a particular site is queried then there there must have been an open path and a closed path from that site to the right boundary. This is known as a two arm event, which for sites far from the boundary gives a revealment of at most $\alpha_2(n) \lesssim n^{-1/4+\varepsilon}$ for any $\varepsilon > 0$ (see [46]). By $\lesssim$ here we mean that there is some constant $c \in (0, \infty)$ such that $\alpha_2(n) \leq cn^{-1/4+\varepsilon}$ for all large n.

Some technical arguments show that the same bound holds for sites close to the boundary, therefore this algorithm on $\mathbb{T}$ gives us that for any $\varepsilon > 0$, $\delta_{f_n} \lesssim n^{-1/4+\varepsilon}$, which decays to zero for small ε. A similar algorithm on $\mathbb{Z}^2$ gives the same revealment with $1/4$ replaced with some other positive constant.

Another benefit of this result is that it gives a criterion for quantitative noise sensitivity. To be precise, if $\delta_{f_n} \lesssim n^{-\beta}$ then for all $\gamma < \beta/2$

$$\lim_{n\to\infty} \mathbb{E}[f_n(\omega)f_n(\omega_{n^{-\gamma}})] - \mathbb{E}[f_n(\omega)]^2 = 0.$$

In the case of $\mathbb{T}$, the above holds for $\beta = 1/4$.

Randomised algorithms were also used to get a lower bound of $1/6$ for Theorem 1.3, which while incredibly important (as it gives us existence of exceptional times) does not provide us with the strongest possible bound. Also, this method did not succeed in proving the existence of exceptional times for $\mathbb{Z}^2$. We now briefly outline the proof of the lower bound.

We define $f_R(t)$ to be the indicator function of the event that there is an open path of length R containing the origin at time t. Ignoring many details for brevity (see [19, Proposition 11.3] - or Section 4.3 for a similar argument), exceptional times exist provided that for all R

$$\mathbb{E}[X_R^2] \leq C\mathbb{E}[X_R]^2$$

where

$$X_R = \int_0^1 f_R(t)\,\mathrm{d}t$$

the Lebesgue measure of the set of times t where there is an open path of length R containing the origin. Fubini's theorem alongside time-invariance (the fact that $f_R(t) = f_R(0)$ in law as the dynamics preserve the distribution of the system) gives us that

$$\mathbb{E}[X_R]^2 = \left(\int_0^1 \mathbb{E}[f_R(t)]\ \mathrm{d}t\right)^2 = \left(\int_0^1 \mathbb{E}[f_R(0)]\ \mathrm{d}t\right)^2 = \mathbb{P}(f_R(0) = 1)^2 = \alpha_1(R)^2$$

the (squared) probability of a one-arm event, the event that there is a single open path from the centre of a ball of radius R to its boundary. By time-invariance

$$\mathbb{E}[X_R^2] \leq 2\int_0^1 \mathbb{P}(f_R(0) \cap f_R(s))\,\mathrm{d}s.$$

Bounding the integrand can be done using some theory relating revealments to the energy spectrum (defined in the next subsection), as well as computing an efficient algorithm for the radial event we are considering here, as opposed to the chordal case we discussed earlier. Omitting a lot of mathematics it is proven that

$$\mathbb{P}(f_R(0) \cap f_R(s)) \lesssim s^{-5/6}\alpha_1(R)^2$$

which integrates to a constant times $\alpha_1(R)^2$ as required. This probability can be multiplied by a factor of $s^{-\gamma}$ for any $\gamma < 1/6$ without the integral diverging, and this fact alongside Frostman's lemma gives us a lower bound of $1/6$ on our Hausdorff dimension.

1.2.4 Method 3: Fourier analysis and spectral sample

This last method, while the most powerful, is also the most abstract. While we will be precise with the definitions and results, we will not really go into any detail regarding the proofs. For that detail, we recommend Chapters 4, 9, 10 and 11 in [19].

We consider $L^2(\Omega_n)$, the vector space of real-valued functions on Ω_n, which Boolean functions are contained in. This space can be equipped with the inner product

$$\langle f, g\rangle := \mathbb{E}[fg] = \sum_{\omega\in\Omega_n} 2^{-n} f(\omega)g(\omega)$$

where the expectation is with respect to the uniform measure. $L^2(\{-1,1\}^n)$ has an orthonormal basis with respect to this inner product given by $\{\chi_S \;:\; S\subset[n]\}$ ($[n] = \{1,\dots,n\}$), where for $\omega = (x_1,\dots,x_n)$

$$\chi_S(\omega) := \prod_{i\in S} x_i.$$

By basic properties of orthonormal bases, we have that for any $f\in L^2(\Omega_n)$

$$f = \sum_{S\subset[n]} \hat{f}(S)\chi_S$$

where the Fourier-Walsh coefficients $\hat{f}(S)$ satisfy

$$\hat{f}(S) = \mathbb{E}[f\chi_S] = \langle f, \chi_S\rangle.$$

Clearly $\hat{f}(\emptyset) = \mathbb{E}[f]$. The set of Fourier-Walsh coefficients is called the spectrum, and analysis of the spectrum gives some indication about the behavior of f, for example whether or not f is noise sensitive. We also define

$$\hat{\mathbb{P}}_f(\{S\})\mathbb{E}[f^2] = \hat{\mathbb{Q}}_f(\{S\}) := \hat{f}(S)^2$$

where $\hat{\mathbb{Q}}_f$ is the spectral measure on the measurable space $(\mathcal{P}([n]),\mathcal{P}(\mathcal{P}([n])))$, where $\mathcal{P}(A)$ is the power set of a set A, while $\hat{\mathbb{P}}_f$ is the spectral probability measure on the measurable space $(\mathcal{P}([n]),\mathcal{P}(\mathcal{P}([n])))$. Note by Parseval's identity (see e.g. [32, Corollary 5.5.4]) we have that

$$\mathbb{E}[f^2] = \sum_{S\subset[n]} \hat{f}(S)^2$$

and if this equals 1 then $\hat{\mathbb{Q}}_f = \hat{\mathbb{P}}_f$. For the rest of this subsection we define our Boolean functions to have codomain $\{-1,1\}$ rather than $\{0,1\}$ so that this indeed occurs. Remember that with the codomain $\{0,1\}$ we can write $f = \mathbb{1}_{A_f}$ so moving into this new codomain corresponds to studying

$$g = \mathbb{1}_{A_f} - \mathbb{1}_{A_f^c}$$

rather than f. Denote $\hat{\mathbb{E}}_f$ as the expectation under $\hat{\mathbb{P}}_f$, and $\mathcal{S}$ for a random subset of $[n]$ generated from $\hat{\mathbb{P}}_f$. To be precise, we sample from $\hat{\mathbb{P}}_f$ conditioned on solely the singleton sets within $\mathcal{P}([n])$. Define $B_S \sim Bin(|S|, \varepsilon/2)$. Using the orthonormality of the (χ_S), alongside that $\chi_S(X)\chi_S(X_\varepsilon) = 1$ iff an even number of bits in S change (this is similar to Example 2 which dealt with the parity function), gives that

$$\begin{aligned}
\mathbb{E}[f(X)f(X_\varepsilon)] - \mathbb{E}[f(X)]^2 &= \sum_{S\subset[n]} \hat{f}(S)^2 \mathbb{E}[\chi_S(X)\chi_S(X_\varepsilon)] - \hat{f}(\emptyset)^2 \\
&= \sum_{S\subset[n]} \hat{f}(S)^2 (\mathbb{P}(B_S \text{ is even}) - \mathbb{P}(B_S \text{ is odd})) - \hat{f}(\emptyset)^2 \\
&= \sum_{S\subset[n]} \hat{f}(S)^2 (1-\varepsilon)^{|S|} - \hat{f}(\emptyset)^2 \\
&= \sum_{m=1}^{n} \left(\sum_{|S|=m} \hat{f}(S)^2 \right) (1-\varepsilon)^m \\
&= \hat{\mathbb{E}}_f[(1-\varepsilon)^{|\mathcal{S}|} \mathbb{1}_{\{\mathcal{S}\neq\emptyset\}}].
\end{aligned}$$

Therefore if we have a sequence (f_n), it is noise sensitive iff $|\mathcal{S}_{f_n}| \to \infty$ in probability, given $\mathcal{S}_{f_n} \neq \emptyset$. In a way this means that noise sensitive functions have high frequencies. On $\mathbb{T}$, [19, Theorem 10.3] states that *approximately*

$$\hat{\mathbb{P}}(0 < |\mathcal{S}_{f_n}| < u) \asymp n^{-1/2} u^{3/4}$$

while [19, Theorem 9.8] says that there exists a constant $c > 0$ such that

$$\hat{\mathbb{P}}(|\mathcal{S}_{f_n}| \geq cn^2 \alpha_4(n)) = \hat{\mathbb{P}}(|\mathcal{S}_{f_n}| \geq cn^{3/4}) \geq c \quad \forall n \in \mathbb{N}$$

which gives us the noise sensitivity asked for in Theorem 1.2. This method also grants us the lower bound of 31/36 for the Hausdorff dimension via a similar bound but on $|\mathcal{S}_{g_n}|$ where g_n is the indicator of the one-arm event on $B_R(0)$ for some fixed $R > 0$. As in the previous subsection, to get the exceptional times results one must focus on the radial events rather than chordal/rectangular events.

1.3 The compass random walk

In this section we introduce the compass random walk, which is the "canonical" simple symmetric random walk probabilists are used to. We also introduce the dynamical compass walk and review key results regarding this process, as well as noise sensitivity results. We split our work by dimension. The last subsection discusses results regarding the compass walk with Gaussian steps.

1.3.1 One dimensional case

Let $X_1, X_2, \ldots$ be independent Rademacher random variables i.e.

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$$

for each $i \in \mathbb{N}$. Define, for each $n \geq 0$,

$$Y_n = \sum_{j=1}^{n} X_j$$

where we take the empty sum to be zero, so $Y_0 = 0$. We call $Y = (Y_n,\, n \geq 0)$ the one dimensional compass random walk. In other words, at each step independently, we jump upwards with probability 1/2 and downwards with probability 1/2. We call this walk the compass walk because there is an inherent sense of direction, $X_i = 1$ always corresponds to an upwards jump etc.

Viewing $Y = Y(X)$ as a function of the step variables, we can use the construction from Section 1.1.3 to construct the dynamical compass random walk $Y(t) = Y(X(t))$, more explicitly written as

$$Y_n(t) = \sum_{j=1}^{n} X_j(t).$$

We can ask whether properties of this walk are noise sensitive or have exceptional times.

Before going into that, it is prudent to think about how the process evolves over time. One way to do this is to consider how the path of the walk changes if only a single step changes its value. If the mth step is the only step to change its value between times 0 and t then:

$$|Y_n(0) - Y_n(t)| = |\sum_{j=1}^{n} X_j(0) - \sum_{j=1}^{n} X_j(t)| = \begin{cases} 2|X_m(0)| = 2 & n \geq m \\ 0 & n < m \end{cases}$$

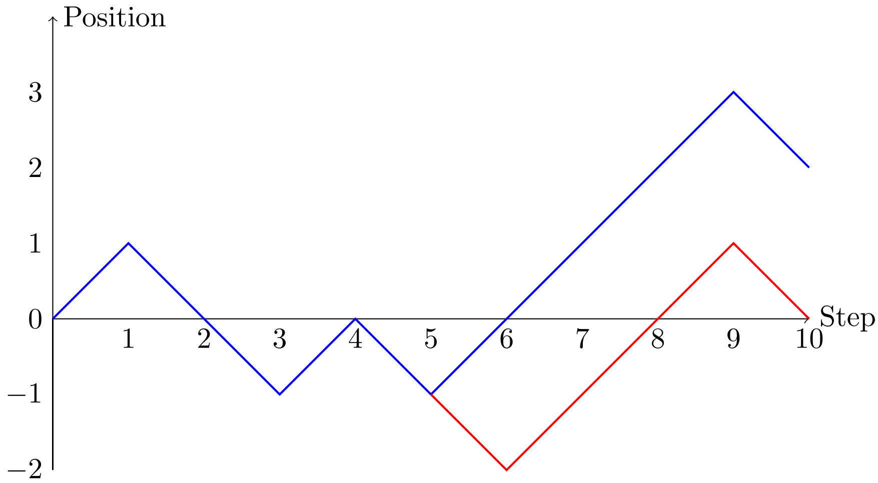

Figure 1-5: A realisation of $(Y_n(0))$ (blue) and $(Y_n(t))$ (red) for $n \leq 10$ where $X_j(0) \neq X_j(t)$ only for $j = 6$.

In other words, once the change is visible, it causes a shift of size two. See Figure 1-5 for an example.

Now consider looking at this path for large n, it would not be unreasonable to think that a single shift of size two is not going to impact the behavior of the walk much. The question then becomes whether or not a chain of these shifts (which could be in either direction) may produce noise sensitive events.

First of all consider the sequence of events indicating whether or not Y_n is positive, that is, $(\{Y_n > 0\}, n \geq 0)$. We claim that this sequence is noise stable. In fact, this is identical to what was proved in Example 3, that the Majority function is noise stable. To be precise, it is identical except for the fact that the Majority function asks whether the sum is non-negative, rather than positive. By symmetry between $(Y_n(t))$ and $(-Y_n(t))$ this slight difference does not matter.

The process $(Y(t))$ has been studied intensively in [7] by Benjamini, Häggström, Peres and Steif, with fewer restrictions on the law of the individual X_j's. We now state their one-dimensional results in full generality. These results are [7, Proposition 1.1, Theorem 1.2 and Theorem 1.11] respectively.

Theorem 1.4. *If $\mathbb{E}[X_1] = \mu < \infty$ then the Strong Law of Large Numbers (SLLN) is dynamically stable for Y. That is,*

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{Y_n(t)}{n} = \mu \ \forall t\right) = 1.$$

Theorem 1.5. *Let $\mathbb{E}[X_1] = 0$ and $Var(X_1) = \sigma^2 < \infty$. Then the Law of the Iterated Logarithm (LIL) is dynamically stable for Y. That is,*

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{Y_n(t)}{\sigma\sqrt{2n\log\log n}} = 1 \ \forall t\right) = 1.$$

Theorem 1.6. *Let X_1 be concentrated on $\mathbb{Z}$, have finite support and $\mathbb{E}[X_1] = 0$. Then recurrence is dynamically stable for Y. That is,*

$$\mathbb{P}\left(\forall t \ Y_n(t) = 0 \textit{ for infinitely many } n\right) = 1.$$

This last result is further strengthened in [25, Theorem 1.5] to mean 0, variance 1 steps that are concentrated on $\mathbb{Z}$, provided there exists $\varepsilon > 0$ such that we have finite absolute $2+\varepsilon$ moments.

So, as expected from the heuristics the compass walk seems to be a very stable process under noise and dynamics. The main draw of this thesis is that we can generate another one-dimensional simple symmetric random walk that is far less stable. This walk will be introduced in the next section.

1.3.2 Two dimensional case

It was conjectured in [7] that unlike in Theorem 1.6, recurrence should be dynamically sensitive in two dimensions. We first define the walk. Let $X_1, X_2, \ldots$ be identically distributed random variables satisfying

$$\mathbb{P}(X_1 = e_1) = \mathbb{P}(X_1 = -e_1) = \mathbb{P}(X_1 = e_2) = \mathbb{P}(X_1 = -e_2) = 1/4$$

where e_1, e_2 are the standard unit vectors in $\mathbb{R}^2$. We then define $Y_0^{(2)} = 0$ and for $n \in \mathbb{N}$

$$Y_n^{(2)} = \sum_{j=1}^{n} X_j$$

so that $(Y_n^{(2)})$ is the two-dimensional compass walk. Again there is an inherent and external sense of direction, in that at each step the walk decides whether to go north, south, east or west.

The intuition behind the recurrence being dynamically sensitive for $(Y_n^{(2)})$ is that it is know that (see e.g. [28, Theorem 4.1.1] and its proof) for any random walk (S_n) we have that S_n is recurrent iff

$$\sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) = \infty.$$

The same reference also states the fact that $\mathbb{P}(Y_n^{(d)} = 0) \asymp n^{-d/2}$ for any dimension $d \in \mathbb{N}$ (the $d \geq 3$ definition of the compass walk is in the next subsection). By $\asymp$ we mean that both $\mathbb{P}(Y_n^{(d)} = 0) \lesssim n^{-d/2}$ and $\mathbb{P}(Y_n^{(d)} = 0) \gtrsim n^{-d/2}$. Using these facts when $d = 2$

$$\sum_{n=1}^{\infty} \mathbb{P}(Y_n^{(2)} = 0) \asymp \sum_{n=1}^{\infty} n^{-1} \asymp \log n \to \infty$$

so we have very slow divergence to infinity, implying that we have "weak" recurrence which could be affected by noise/dynamics.

This conjecture was proven true by Hoffman [23], who not only showed that exceptional times of transience exist almost surely, but that the Hausdorff dimension of the set of such times is 1 almost surely. Hoffman and Amir [3] then showed that almost surely there were times where the origin was the only position to be visited finitely many times.

1.3.3 Dimensions three and higher

While Benjamini et al in [7] did not cover the $d = 2$ case, they did in fact fully analyse $(Y_n^{(d)})$ for all $d \geq 3$. Their quite general construction is as follows:

Take a discrete Abelian group G with identity element 0 and then define a symmetric probability measure ν on G (so that $\nu(g) = \nu(-g)\ \forall g \in G$). From there define the processes $X_n(t)$ to be dynamical random variables that rerandomise over time as in Section 1.1.3 but with respect to ν, and then define

$$S_n(t) = \sum_{i=1}^{n} X_i(t)$$

as usual. In our particular setting, we have $G = \mathbb{Z}^d$ and $S = Y^{(d)}$ (for any $d \in \mathbb{N}$) under the usual component wise addition, and ν defined such that $\nu(e_i) = \nu(-e_i) = 1/2d\ \forall i = 1, \ldots, d$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^\intercal$ is the ith standard basis vector of $\mathbb{Z}^d$. Note that this construction matches up to our model in $d = 1, 2$.

For $d \geq 3$, simple symmetric random walks are transient by [28, Theorem 4.1.1], so it is exceptional times of recurrence that we are searching for. Indeed Benjamini et al found a criterion for the existence of such times in [7, Theorem 5.5]:

Theorem 1.7. *For a dynamical random walk $(S_n(t))_{n,t}$ defined as above, we have that*

$$\mathbb{P}(\exists t \ : \ S_n(t) = 0 \ i.o.) = \begin{cases} 0 & if \ \sum_{n=0}^{\infty} n\mathbb{P}(S_n = 0) < \infty \\ 1 & if \ \sum_{n=0}^{\infty} n\mathbb{P}(S_n = 0) = \infty \end{cases}$$

Again, $\mathbb{P}(Y_n^{(d)} = 0) \asymp n^{-d/2}$ so exceptional times of recurrence do not exist for $d \geq 5$, that is, transience is dynamically stable. While for $d \in \{3, 4\}$ transience is dynamical sensitive and the Hausdorff dimension of the set of exceptional times is $(4 - d)/2$ by [7, Theorem 1.13]. Note in particular that for $d = 4$ the set is a.s. non-empty despite having Hausdorff dimension and Lebesgue measure 0.

1.3.4 The compass walk with Gaussian steps

We return to the one dimensional dynamical compass random walk $(Y_n(t))$, but now the step variables $(X_j(t))$ are standard normal $N(0, 1)$ random variables. This case was studied by Khoshnevisan et al in [25, 26].

We say a function $f(n)$ is in the upper class of $Y(t)$ if $Y_n(t) \leq f(n)$ for all but finitely many n.

A celebrated result by Erdős [16] is that for a non-decreasing function $H : \mathbb{R}^+ \to \mathbb{R}^+$, $H(n)\sqrt{n}$ is in the upper class of $Y(0)$ iff

$$\int_1^\infty H^2(x)(1 - \Phi(H(x)))x^{-1} \, \mathrm{d}x < \infty$$

where Φ is the standard normal CDF. [26, Theorem 1.5] by Khoshnevisan et al says that we almost surely have exceptional times where $H(n)\sqrt{n}$ is not in the upper class of $Y(t)$ iff

$$\int_1^\infty H^4(x)(1 - \Phi(H(x)))x^{-1} \, \mathrm{d}x = \infty.$$

[26, Corollary 1.7] gives an explicit example of this difference at work. In that

$$\limsup_{n\to\infty} \frac{Y_n^2(0) - 2n \log\log n}{n \log\log\log n} = 3 \ a.s.$$

yet there almost surely exists a time t where the above equals 5 almost surely.

Khoshnevisan et al furthered their own results in [25, Theorem 1.1] by giving us the

Hausdorff dimension of the exceptional times (if any). That being

$$\min\left(1, 2 - \frac{1}{2}\delta(H)\right)$$

where a negative Hausdorff dimension corresponds to there being no exceptional times almost surely, and

$$\delta(H) := \sup\left\{\gamma > 0 \ : \ \int_1^\infty H^\gamma(x)(1 - \Phi(H(x)))x^{-1}\,\mathrm{d}x < \infty\right\}.$$

Other results are proven. Define for $s, t \in [0, 1]$,

$$U_n(s, t) = \frac{1}{\sqrt{n}} Y_{\lfloor ns \rfloor}(t)$$

then we have that by [26, Theorem 1.1]: $(U_n(s, t))_{s,t}$ converges weakly in $D([0, 1]^2)$ (Skorohod space) as $n \to \infty$ to the process $(U(s, t))_{s,t}$ which is a continuous centred Gaussian field with covariance

$$\mathbb{E}[U(s_1, t_1)U(s_2, t_2)] = \min(s_1, s_2)e^{-|t_1 - t_2|} \ \forall s_1, s_2, t_1, t_2 \in [0, 1].$$

In particular, for every fixed s, $(U(s, t))_t$ is an Ornstein-Uhlenbeck process, and of course if t is fixed then $(U(s, t))_s$ is a Brownian motion by Donsker's Invariance Principle.

This was strengthened in [25, Theorem 1.4] where this convergence was proven to occur for any step distribution of mean zero and variance one, which includes the Rademacher model that we are interested in.

1.4 The switch random walk

Here we introduce the switch random walk, which is our main object of study in Chapter 2. A variation of it is also studied in Chapter 3.

1.4.1 Definition and comparisons to the compass walk

As in the previous section, let $X_1, X_2, \ldots$ be independent Rademacher random variables, i.e.

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$$

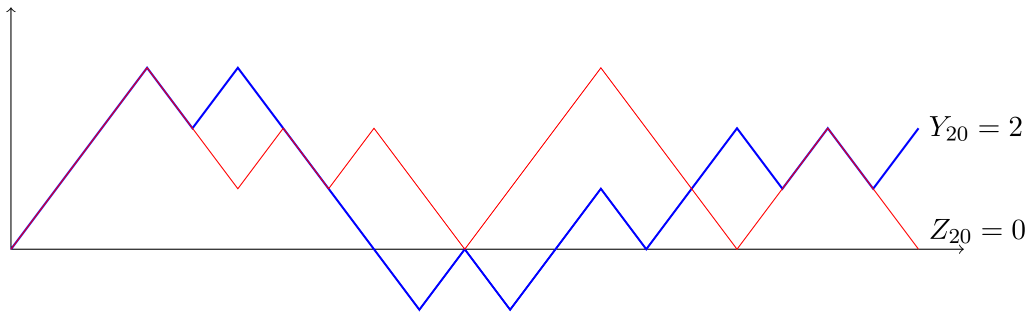

Figure 1-6: First 20 steps of the Compass (blue) and Switch (red) random walks for the vector $X = (1, 1, 1, -1, 1, -1, -1, -1, -1, 1, -1, , 1, 1, -1, 1, 1, -1, 1, -1, 1)$.

for each $i \in \mathbb{N}$. Define, for each $n \geq 0$,

$$Z_n = \sum_{k=1}^{n} \prod_{j=1}^{k} X_j$$

where again we take $Z_0 = 0$. We call $Z = (Z_n, n \geq 0)$ the switch random walk. We can think of Z as a function of the step variables, $Z = Z(X)$.

To explain this process intuitively, at each step independently, with probability $1/2$ we take a step in the direction we are currently facing, and with probability $1/2$ we turn around and take a step. Our initial orientation at step 0 is facing upwards.

We call (Z_n) the switch random walk as there is no external sense of direction, it is purely based on the view of the walker itself.

It is easy to see that, although they are different functions, the two walks Y and Z have the same distribution, see Figure 1-6 for any example of a specific realisation. However, Z is more sensitive to changes in the sequence X, in a sense that we will make precise below. We can define the dynamical switch random walk $Z(t) = Z(X(t))$ in the usual way, i.e.

$$Z_n(t) = \sum_{k=1}^{n} \prod_{j=1}^{k} X_j(t).$$

To see why this walk might be more sensitive to small changes, let's again see what happens if the mth step is the only step to differ between times 0 and t:

$$|Z_n(0) - Z_n(t)| = |\sum_{k=1}^{n} \prod_{j=1}^{k} X_j(0) - \sum_{k=1}^{n} \prod_{j=1}^{k} X_j(t)| = \begin{cases} 2|\sum_{k=m}^{n} \prod_{j=1}^{k} X_j(0)| & n \geq m \\ 0 & n < m \end{cases}$$

Now for m fixed but n large, the expression on the right hand side is just 2 times

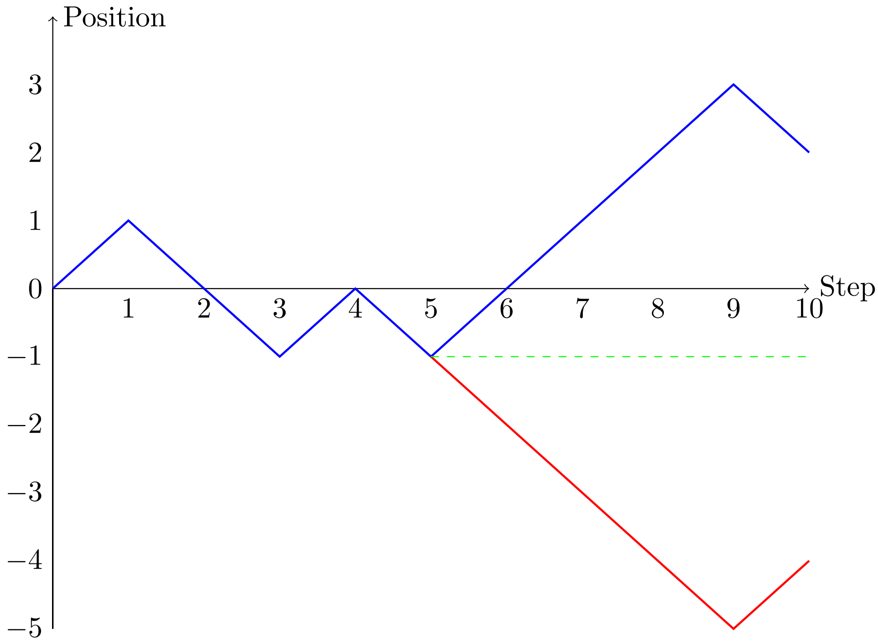

Figure 1-7: A realisation of $(Z_n(0))$ (blue) and $(Z_n(t))$ (red) for $n \leq 10$ where $X_j(0) \neq X_j(t)$ only for $j = 6$. The green marks the line of reflection.

the modulus of a simple symmetric random walk, the expectation of which grows like $\sqrt{n}$. An explicit derivation of this fact can be found in [37]. Of course, this is far more dramatic than a shift of two as in the compass case. Also note that this shift corresponds to the path being reflected from the value of the walk at Z_{m-1}, as the cumulative product at time t has the opposite sign as the time 0 product for all $k \geq m$. See Figure 1-7 for an example of this.

1.4.2 The coin-turning walk

The object that we refer to as the switch random walk is also known by other names. In this subsection we cover some of the research that has been done on (generalisations of) the switch random walk (Z_n). Nothing in this subsection is required for the rest of the thesis, however it is all still very interesting.

The switch random walk has been called the coin-turning random walk by Engländer and Volkov who introduced more general (static) versions in [14], and these were further studied by Engländer, Volkov and Wang [15].

The walk they studied, which we call $(Z_n^{(\tilde{p})})$, is defined via a sequence of numbers $\tilde{p} = (p_n)_{n\in\mathbb{N}}$ in $[0, 1]$ where p_n corresponds to the probability that the walk changes direction relative to its current direction on the nth step. In particular our walk (Z_n) is the case where $p_n = 1/2$ for all n. However they work with varying probabilities, which can even depend on n.

The reason they call the walk the coin-turning walk is because they interpret the p_n's as the probability that you turn the coin over on the nth step, where heads is $+1$ and tails is -1. This is except for the case where $n = 1$, p_1 is a flipping probability, as the coin is flipped to start the process. Of course, not turning a coin over is equivalent to moving in the same direction that you were already travelling.

It is worth pausing at this point to mention that the coin-turning walk can also be viewed as a special case of the Gillis-Domb-Fischer "correlated random walk". In particular, the case that is in fact uncorrelated. See [20, 41] for references, which we will not discuss here, but will mention again in Section 5.1.3.

In [14], Engländer and Volkov focus primarily on Central limit theorem results, as well as the proportion of heads achieved during the entire process. A summary of these results is as follows:

- If $\sum_{n=1}^{\infty} p_n < \infty$ then $\frac{1}{n} Z_n^{(\tilde{p})} \to \pm 1$ with probability $1/2$ each.
- If $p_n = c \in (0, 1)$ $\forall n$, then
$$\frac{1}{\sqrt{n}} Z_n^{(\tilde{p})} \to N(0, (1 - c)/c) \text{ in law.}$$
- Let $a > 0$ and assume $p_n = a/n$ for all large n, then
$$\frac{1}{n} Z_n^{(\tilde{p})} \to Beta(a, a) \text{ in law.}$$
- Let $a > 0, \gamma \in (0, 1)$ and assume $p_n = a/n^\gamma$ for all large n. Then
$$\frac{1}{n} Z_n^{(\tilde{p})} \to N(0, \sigma_{a,\gamma}^2) \text{ in law, where } \sigma_{a,\gamma}^2 = \frac{1}{a(1+\gamma)}.$$

Note that our switch walk satisfies the second bullet point where $c = 1/2$, giving $(1 - c)/c = 1$, so indeed we satisfy the usual CLT, as expected. In [15], focus turns to scaling limits of the coin turning walk in multiple regimes, but mainly the heating regime ($p_n \to 1$) and the cooling regime ($p_n \to 0$). Before discussing those two cases, it was proven that if $p_n = c \in (0,1)$ for all large n then

$$\left(\frac{1}{\sqrt{n}} Z^{(\tilde{p})}_{\lfloor cnt/(1-c)\rfloor}\right) \to (B_t) \text{ in law.}$$

This again is the case that (Z_n) falls into. In the heating case, it is shown [15, Theorem 3] that under many technical assumptions that we omit,

$$\left(\frac{1}{\sqrt{n}} Z^{(\tilde{p})}_{T(nt)}\right) \to (B_t) \text{ in law}$$

where

$$T(x) = \inf\left\{ n \in \mathbb{N} \;:\; \sum_{i=1}^{n} 4p_i(1-p_i)\left(\sum_{j=1}^{\infty} Cov(\prod_{k=1}^{i} X_k, \prod_{k=1}^{i+j} X_k)\right)^2 \geq x \right\}.$$

In short, we get a Brownian motion provided we scale time differently to normal. In the cooling case, the above holds provided $p_1 = 1/2$, $p_n \to 0$ and $np_n \to \infty$. When the latter condition fails we get quite different scaling limits, and the interested reader should read [15, Theorem 4].

1.4.3 The bootstrap random walk

Our walk (Z_n) has also been called the bootstrap random walk by Collevecchio, Hamza and Shi in [13]. The motivation behind calling (Z_n) the bootstrap random walk is that to generate (Z_n) we reuse the increments of (Y_n), so echoes the resampling method of bootstrapping.

As previously discussed, (Y_n) and (Z_n) have the same law, but they are evidently not independent. Collevecchio, Hamza and Shi show that this dependence is lost in the limit, in that as $n \to \infty$

$$\left(\frac{1}{\sqrt{n}}(Y_{\lfloor nt\rfloor}, Z_{\lfloor nt\rfloor})\right)_t$$

converges to a two-dimensional Brownian motion with independent components. They actually prove that this holds in $d \in \mathbb{N}$ dimensions if you look at successive bootstrapped walks. For example the next walk in the sequence would be $S = (S_n)$ where

$$S_n = \sum_{k=1}^{n} \prod_{j=1}^{k} \prod_{i=1}^{j} X_i$$

and that the scaling limit of (Y, Z, S) would be a three dimensional Brownian motion with independent components.

Collevecchio, Hamza and Liu gave a further generalisation of the bootstrapped random walk in [12]. This generalisation is as follows:

1. Generate a sequence of Rademacher random variables, $(X_n^{(0)})$, and fix $K \in \mathbb{N}$.
2. For $0 \leq i < K$ and $j \geq 0$ define

$$X_j^{(i+1)} = \prod_{k=1}^{j} X_k^{(i)}.$$

3. For $0 \leq i < K$ and $j \geq 0$ define:

$$X_j^{(-i+1)} X_{j-1}^{(-i+1)} = X_j^{(-i)}.$$

The rules governing filling up this entire tableau of values given just one row is known as the "cellular automata rules". Defining $(Z_n^{(i)})$ as the cumulative row sum of the ith row, it is shown [12, Theorem 2.2] that the collection of $2K+1$ walks has a scaling limit of $2K+1$ dimensional Brownian motion with a non-standard covariance matrix. It is also proven that if we restrict ourselves to the $K+1$ non-negatively indexed walks then we have a scaling limit to $K+1$ dimensional standard Brownian motion. [12, Theorem 2.1] states that any d-dimensional projection of d of the non-negatively indexed walks has the same recurrence/transience properties as the standard d-dimensional simple symmetric random walk.

1.5 Warren's random walk

The moral of the story of Chapter 2 will be that while the compass walk (Y_n) and the switch walk (Z_n) have the same law as a sequence, they exhibit different properties when exposed to noise, or when evolving over time. At the time of discovering this result, we believed this to be the first known example of two sequences with this relationship. However, it was brought to our attention by Jon Warren that he in fact found a different random walk (W_n) that was also sensitive to noise. This section is dedicated to discussing the random walk that Warren analysed in [51].

It is worth emphasising at this point that we do not utilise Warren's walk elsewhere in the thesis, and that this section is a discussion and so not every piece of mathematics will be fully rigorous.

1.5.1 Motivation and result

Warren was interested in understanding some ideas produced by Tsirelson in [45, 49, 50] (the former being joint with Schramm) by applying said ideas to Tanaka's SDE

$$\mathrm{d}W_t = \operatorname{sign}(W_t)\,\mathrm{d}B_t$$

where B_t is a standard Brownian motion. This is otherwise written

$$W_t = \int_0^t \operatorname{sign}(W_s)\,\mathrm{d}B_s$$

and has a weakly unique solution but no strong solution. It can be shown that

$$|W_t| = B_t + \sup_{s\le t}(-B_s).$$

In particular, $B = (B_s)_{s\le t}$ determines the magnitude of W_t, but it does not uniquely define the path of $W = (W_t)_t$ itself.

Warren then studied the discrete version of the solution to Tanaka's SDE, that is, the process $W = (W_n)_{n\ge 0}$ where $W_0 = 0$ and

$$W_n := \sum_{k=1}^{n} \operatorname{sign}(W_{k-1})X_k = \sum_{k=1}^{n} \operatorname{sign}(W_{k-1})(Y_k - Y_{k-1})$$

where all the X_k are IID Rademacher random variables, $Y = (Y_k)$ is the compass walk from Section 1.3.1, and $\operatorname{sign}(W_{k-1}) = 1$ if W_{k-1} is non-negative and -1 otherwise. This is a simple, symmetric random walk as W_{k-1} is independent of X_k.

We denote both the discrete and continuous model by W, it should always be clear from context which is which.

The rule this walks follows is "if W_n is currently non-negative then to get to W_{n+1} you should move as X_{n+1} tells you to move, but if you're negative then do the opposite of X_{n+1}". See Figure 1-8 for an example of such a walk. In the next subsection we shall discuss this in more detail.

Warren observes that in the discrete case the compass walk Y determines the precise path of W, while in the continuous case information regarding the sign of W is lost. In other words, this information is lost in the scaling limit.

As Warren states, an explanation for the sign being lost in the limit is due to the sign

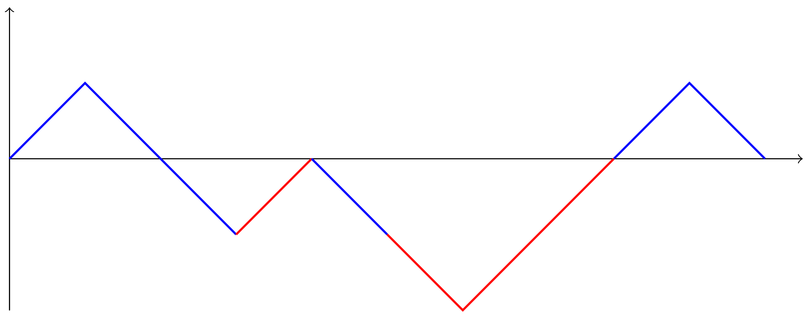

Figure 1-8: An example of the first 10 steps Warren's walk with the step vector $X = (1, -1, -1, -1, -1, 1, -1, -1, 1, -1)$. In red are the steps where the walk does the opposite of what X tells it to (if the walk were the compass walk Y).

of W being noise sensitive. The aim of this section is to flesh out the noise sensitivity arguments presented by Warren, in the following theorem:

Theorem 1.8. *$(\{W_n \geq 0\}, n \geq 1)$ is noise-sensitive. That is for any $t > 0$*

$$\mathbb{P}(W_n(0) \geq 0,\ W_n(t) \geq 0) - \mathbb{P}(W_n(0) \geq 0)^2 \to 0$$

where $W(t) = (W_n(t))$ is defined by using the dynamical random variables $X_k(t)$ as in Section 1.1.3.

This result parallels Theorem 2.2 for (Z_n) in the specific case where $\varepsilon_n = t\ \forall n$. By symmetry between $(Z_n(t))$ and $(-Z_n(t))$ it is irrelevant whether or not we have $\geq$ or $>$ in the theorem.

1.5.2 Behavior of Warren's walk

Before sketching the proof of Theorem 1.8, we shall try to understand the behaviour of $W = W(0)$, and then move on to the dynamical case. A good way to analyse W is to study Y which is the compass random walk from Section 1.3.1 as it turns out that they are highly related.

We list the relationship between W and Y fully in the following proposition:

Proposition 1.9. *Define $I'_0 = 0$ and $I'_k = \inf\{n > I'_{k-1}\ :\ Y_n = -k\}$, the step that Y attains a new minimum since its previous minimum. For $k \in \mathbb{N}$ we have the following:*

- *When k is odd: $I'_k = \inf\{n > I'_{k-1}\ :\ W_n = -1\}$, and for $n \in (I'_{k-1}, I'_k]$ we have that W and Y move in the same direction, in particular $|Y_n - W_n| = k$ in this period. Also W is non-negative in $[I'_{k-1}, I'_k)$.*

- *When k is even: $I'_k = \inf\{n > I'_{k-1} \ : \ W_n = 0\}$, and for $n \in (I'_{k-1}, I'_k]$ we have that W and Y move in the opposite direction. Also W is negative in $[I'_{k-1}, I'_k)$.*

To see what this Proposition looks like in practice, see Figure 1-9 and keep it in mind when reading the proof.

Proof of Proposition 1.9. We prove this by induction, starting the with base cases of $k = 1, 2$. Two base cases aren't necessary, but it helps to show the argument in both the odd and even case.

For $k = 1$, we have $Y_{I'_1} = -1$ and I'_1 is the first step that Y hits -1, ergo the first step that Y is negative. As $W_0 = Y_0 = 0$ we have that $\text{sign}(W_0) = 1$ so W starts off following the path of Y, but Y is non-negative until step I'_1, so $\text{sign}(W) = 1$ throughout, thus $W = Y$ up until I'_1 so the claim clearly holds for $k = 1$.

For $k = 2$ we have that $W_{I'_1} = Y_{I'_1} = -1$, $Y_{I'_2} = -2$ (and is the earliest step we hit -2). Note that through $[I'_1, I'_2)$ the walk Y must have always have at least as many up $(+1)$ steps as down (-1) steps, otherwise it would hit -2 before step I'_2. As $\text{sign}(W_{I'_1}) = -1$, the previous sentence implies that W indeed moves in the opposite direction to Y throughout, so has more down steps than ups, so remains negative. Then on step I'_2 Y we have 1 more down than up for Y, so 1 more up than down for W, meaning that $W_{I'_2} = 0$.

For general k (given the statements are true for $k - 1$) the proof follows the exact same argument as these base cases above. This is because each period $[I'_{k-1}, I'_k)$ is independent so we can use the insight that the first time *#downs* $\geq$ *#ups* for Y in $(I'_{k-1}, I'_k]$ is on step I'_k itself. This is all that is required to prove the proposition. $\square$

On top of the Proposition we have just proved, note the obvious fact that since Y is a random walk that makes steps of size 1, I'_k has to be odd if k is odd and even if k is even. However this observation is very useful as we can combine it with Proposition 1.9 to deduce the following (this was also observed by Warren):

If we know the path of Y up to the nth step and we want to know whether $W_n \geq 0$ or not, then all we have to do is find the last step $r \in \{0, 1, \ldots, n\}$ such that $Y_r = \inf_{k \leq r}(Y_k)$, i.e. find the largest r such that $I'_r \leq n$. If r is even then $W_n \geq 0$ while if it is odd then $W_n < 0$. This immediately translates the event concerning Theorem 1.8 into an event regarding the compass walk Y.

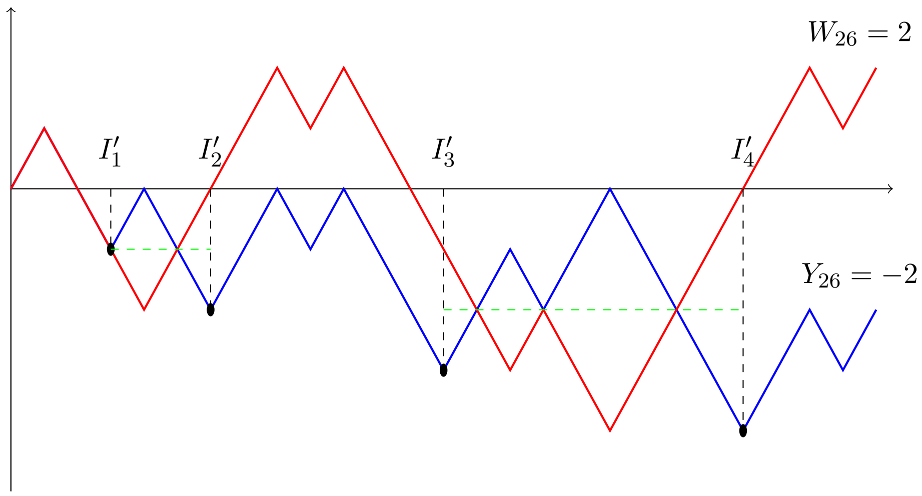

Figure 1-9: First 26 steps of an example of Warren's (red) walk coupled with a Compass (blue) walk.

1.5.3 Sketch proof of Theorem 1.8

Fix $t > 0$. We have that

$$\mathbb{E}[\text{sign}(W_n(0))] = \mathbb{P}(W_n(0) \geq 0) - \mathbb{P}(W_n(0) < 0) = \mathbb{P}(W_n(0) = 0)$$

where we have used that (W_n) is symmetric. Now as $\mathbb{P}(W_n(0) = 0) \to 0$, to prove noise sensitivity it suffices to prove that $\mathbb{E}[\text{sign}(W_n(0))\text{sign}(W_n(t))] \to 0$. For a random walk $S = (S_k)_{0 \leq k \leq n}$ let $M_S = \min\{m \in \mathbb{N} : S_m = \min_{0 \leq k \leq n} S_k\}$, i.e. the first step that the random walk S attains its local minimum over the first n steps. Now

$$\begin{aligned}\mathbb{E}[\text{sign}(W_n(0))\text{sign}(W_n(t))] =& \mathbb{E}[\text{sign}(W_n(0))\text{sign}(W_n(t))\mathbb{1}_{\{M_Y = M_{Y(t)}\}}] \\ &+ \mathbb{E}[\text{sign}(W_n(0))\text{sign}(W_n(t))\mathbb{1}_{\{M_Y \neq M_{Y(t)}\}}] \\ &\to \lim_{n\to\infty} \mathbb{E}[\mathbb{1}_{\{M_Y = M_{Y(t)}\}}] + 0.\end{aligned}$$

This is because if $\{M_Y = M_{Y(t)}\}$ occurs then by the discussion following Proposition 1.9 the signs of $W_n(0)$ and $W_n(t)$ must be the same so sign($W_n(0)$)sign($W_n(t)$)=1. While if $\{M_Y \neq M_{Y(t)}\}$ occurs then the two walks do not attain their minimum simultaneously, so in the limit as $n \to \infty$ there is probability 1/2 that the attainment step of both walks are both even/odd, and probability 1/2 that one is even and the other is odd. By Proposition 1.9 and what followed it, we have that conditioned on $\{M_Y \neq M_{Y(t)}\}$ we have that $\lim_n \text{sign}(W_n(0))\text{sign}(W_n(t))$ is 1 with probability 1/2 and

-1 with probability $1/2$, so the expectation is zero. This means it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}[\mathbb{1}_{\{M_Y = M_{Y(t)}\}}] = 0.$$

Consider the two dimensional walk $((Y_k(0), Y_k(t)),\ 0 \le k \le n)$. We can take the scaling limit to obtain the pair of one-dimensional Brownian motions $((B_s, B'_s),\ 0 \le s \le 1)$ which satisfy

$$dB_s dB'_s = e^{-t} ds.$$

We now define M and M' to be the (random) times in $[0, 1]$ where (B_s) and (B'_s) respectively attain their minimums. By careful manipulation of the limits it can be shown that

$$\lim_{n\to\infty} \mathbb{E}[\mathbb{1}_{\{M_Y = M_{Y(t)}\}}] = \mathbb{E}[\mathbb{1}_{\{M = M'\}}] = \mathbb{P}(M = M').$$

As $t > 0$, we have two Brownian motions that are positively, but not perfectly, correlated. Therefore they do not achieve their minimums at the same time a.s., as required.

1.6 Bramson's branching random walk

We now introduce the final model that we shall be studying, the branching random walk studied by Bramson [11]. We shall mostly discuss the specific case we will be studying in Chapter 4, but we shall also briefly celebrate Bramson's results in full generality.

We define the branching random walk (BRW) on $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as follows. In generation zero we start with one particle at position 0. At each generation, every particle currently alive has 2 children (and the particle dies) and then each of these two children independently either stays where the parent was or jumps to the right by one step, each with probability $1/2$.

We can also construct this BRW using a *deterministic* tree, the binary tree to be exact. We can also view this tree as a Galton-Watson tree (Z_n) with offspring distribution L satisfying $L = 2$ almost surely. Let r denote the root, then define $r_{i_1 \dots i_n}$ to be the node that comes from the $(i_1 + 1)$th child of r, the $(i_2 + 1)$th child of r_{i_1} and so on. Clearly i_1, i_2 etc. above can only take values 0 and 1. We now equip the edges of this tree with random variables $X(r_{i_1 \dots i_n})$ (also written $X_{r_{i_1 \dots i_n}}$ or $X_{i_1 \dots i_n}$) all with distribution X, which is a random variable taking values 1 and 0 with probability $1/2$ each. For

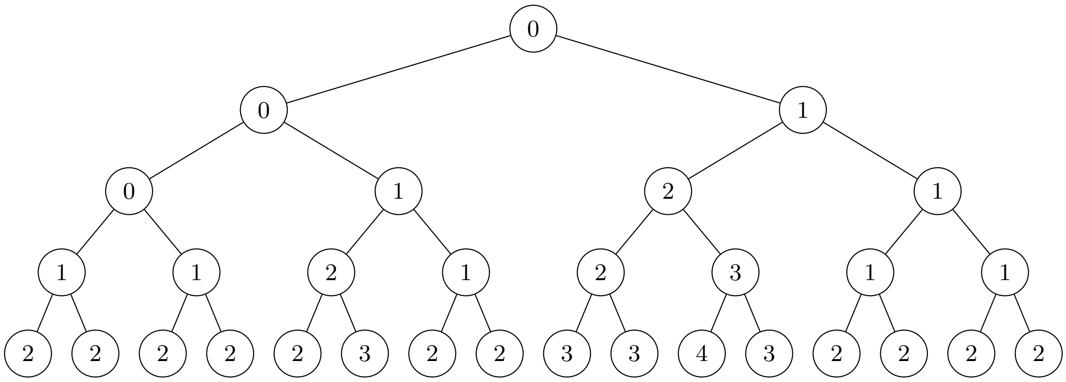

Figure 1-10: A realisation of the branching random walk up until the fourth generation. The value inside a vertex v corresponds to S_v. We have $M_0 = M_1 = M_2 = 0$ and $M_3 = 1$ and $M_4 = 2$.

example we equip the random variables X_0 and X_1 to the edges connecting r_0 to r, and r_1 to r, respectively.

These two descriptions coincide because to see the position of particle $r_{i_1 \ldots i_n}$ we compute

$$S_{r_{i_1 \ldots i_n}} = \sum_{k=1}^{n} X_{r_{i_1 \ldots i_k}}$$

and the fact the tree is binary represents each particle having exactly 2 children, and the distribution of the X's correspond to the jumping to the right with probability $1/2$. See Figure 1-10 for an example of a branching random walk of this kind.

We define the minimum displacement at generation n to be the spatial position of the particle born in generation n that is furthest to the left, i.e.

$$M_n = \min_{i_1 \ldots i_n} S_{r_{i_1 \ldots i_n}}.$$

Bramson shows that

$$M_n \sim \frac{1}{\log 2} \log \log n \ a.s.$$

To be precise, we shall introduce Bramson's main result in (almost) full generality. Bramson considered a far more general model to the one just above. Firstly, he had made no requirement for the tree to be deterministic. He considered a Galton-Watson tree (Z_n) with offspring distribution L that is actually random. He also allowed for more flexibility in that for fixed $p \in (0, 1)$,

$$\mathbb{P}(X = 0) = p = 1 - \mathbb{P}(X = 1).$$

He proved the following theorem, [11, Theorem 1]:

Theorem 1.10. *Assume for some* $\varepsilon > 0$ *that* $\mathbb{E}[L^{2+\varepsilon}] < \infty$, $\mathbb{E}[L] > 1$, *and that* $p\mathbb{E}[L] = 1$. *Then, conditioned on non-extinction of* (Z_n),

$$\lim_{n\to\infty}\left(M_n - \left\lceil \frac{1}{\log 2}(\log\log n - \log(V + o(1)))\right\rceil\right) = 0 \quad a.s.$$

where V *is a non-degenerate random variable, and* $o(1)$ *is stochastic.*

In the situation previously considered, we have $\mathbb{E}[L] = 2$ and $p = 1/2$, hence this theorem is satisfied.

Bramson in fact studies the even more general case where each particle can jump to the right by any (random) quantity in $(0, \infty)$ at each step, in that $\mathbb{P}(X \neq 0) = 1 - p$, but we shall not consider this case here.

The key idea behind the proof of Theorem 1.10 is that of "dynasties". We define the dynasty of a node $r_{i_1\ldots i_k}$ to be $S_{r_{i_1\ldots i_n}}$, its spatial position. Note that the tree consisting of all the nodes in the 0th dynasty, T_0, is the connected component of 0s containing the root r, so is a tree. However the 1st dynasty consists of all nodes v where $S_v = 1$, which is in fact a forest of smaller trees. The number of trees in said forest is in fact the size of the "forward edge boundary" of T_0. Referring back to Figure 1-10, here we see that the 0th dynasty is a single component of size three, while the 1st dynasty consists of four components of sizes $1, 1, 2$ and 4 respectively.

When we return to this model in Chapter 4, we shall make all of the edge variables dynamical as in Section 1.1.3. We will then study the quantity $(M_n(t))$ and present a conjecture stating that exceptional times exist where the left-most particle is significantly further left than as seen in Theorem 1.10. In other words, the asymptotic position of M_n is dynamically sensitive.

1.7 Structure of the thesis

The structure of the rest of this thesis is as follows. Chapter 2, which is based on a paper published jointly with Matthew Roberts [40] focuses on the switch random walk (Z_n). This walk when viewed as a dynamical process exhibits exceptional times of recurrence and noise sensitivity while the compass walk (Y_n) does not.

In the third chapter, we prove the continuous analogue of all the results in chapter two. That is, we study a process called dynamical Brownian motion and show that under our dynamics it exhibits noise sensitive behavior. The main appeal of the results in this chapter is that moving from a discrete process to a continuous one requires developing some new tools. The construction of the dynamics itself is also interesting and could be used to study other models

The fourth chapter revolves around branching random walks, in particular we focus our attention on the position of the left-most particle. As discussed in the previous subsection, Bramson [11] has found the speed at which the left-most particle travels to infinity for a class of branching random walks. We select a particular walk within that class and conjecture deviation probabilities for the position of the left-most particle. We also construct the dynamical version of our example and conjecture that exceptional times exist where the left-most particle travels slower that expected asymptotically.

The fifth and final chapter contains some interesting open questions.

Chapter 2

Exceptional Times Of The Switch Random Walk

In this chapter, we reintroduce (Z_n), a dynamical simple symmetric random walk in one dimension, and show that there almost surely exist exceptional times at which the walk tends to infinity. This is in contrast to the usual dynamical simple symmetric random walk in one dimension, (Y_n), for which such exceptional times are known not to exist. In fact we show that the set of exceptional times has Hausdorff dimension 1/2 almost surely, and give bounds on the rate at which the walk diverges at such times.

We also show noise sensitivity of the event that (Z_n) is positive after n steps. In fact this event is maximally noise sensitive, in the sense that it is quantitatively noise sensitive for any sequence ε_n such that $n\varepsilon_n \to \infty$. This is again in contrast to the usual random walk, for which the corresponding event is known to be noise stable.

This work is joint with Matthew Roberts and appears in [40], albeit the presentation here is slightly different and some additional commentary is included. Due to this there will be some repetition of material from the previous chapter.

2.1 Introduction and results

We remind ourselves of the two one dimensional simple symmetric random walks of interest, the compass and switch random walks. The first, at each step independently, jumps upwards with probability 1/2 or downwards with probability 1/2. The second begins facing upwards and, at each step independently, decides to keep moving the same way with probability 1/2 or switches direction with probability 1/2.

We call the first of these two random walks the *compass* random walk, as it has an in-built sense of direction, and the second the *switch* random walk, as it only decides whether or not to switch directions. Of course these two random walks have exactly the same distribution—they are simple symmetric random walks—although, as we will see when we define them rigorously, they are different functions of the underlying randomness. This means that when we talk about noise sensitivity or dynamical sensitivity of the two walks, they may (and do) have very different properties.

We now define carefully the objects of interest. Let $X_1, X_2, \ldots$ be independent random variables satisfying

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$$

for each $i \in \mathbb{N}$. Define, for each $n \geq 0$,

$$Y_n = \sum_{j=1}^{n} X_j$$

and

$$Z_n = \sum_{k=1}^{n} \prod_{j=1}^{k} X_j$$

where we take the empty sum to be zero, so $Y_0 = Z_0 = 0$. We call $Y = (Y_n,\, n \geq 0)$ the compass random walk, and $Z = (Z_n,\, n \geq 0)$ the switch random walk. We can think of $Y = Y(X)$ and $Z = Z(X)$ as functions of the sequence of random variables $X = (X_1, X_2, \ldots)$. It is easy to see that, although they are different functions, the two walks Y and Z have the same distribution. To be explicit, we have that $Y_{n+1} - Y_n = X_{n+1}$, while $Z_{n+1} - Z_n = \prod_{j=1}^{n+1} X_j$. Indeed, the written descriptions at the beginning of this section make clear that each of the two walks is a natural one-dimensional interpretation of the ant in the labyrinth or the drunkard's walk. However, Z is more sensitive to changes in the sequence X, in a sense that we will make precise below.

We now introduce dynamical versions of our random walks Y and Z, using the dynamics from Section 1.1.3. For each $j \geq 1$, let $(N_j(t), t \geq 0)$ be an independent Poisson process of rate 1, and for each $i \geq 0$, let X_j^i be an independent random variable with $\mathbb{P}(X_j^i = 1) = \mathbb{P}(X_j^i = -1) = 1/2$. Then define

$$X_j(t) = X_j^i \quad \text{whenever} \quad N_j(t) = i.$$

In words, $X_j(t)$ has the same distribution as X_j and rerandomises itself at the times of the Poisson process $N_j(t)$. Write $Y(t) = Y(X(t))$ and $Z(t) = Z(X(t))$, or more explicitly

$$Y_n(t) = \sum_{j=1}^{n} X_j(t) \quad \text{and} \quad Z_n(t) = \sum_{k=1}^{n} \prod_{j=1}^{k} X_j(t)$$

for each $n \geq 0$.

For each fixed $t \geq 0$, both $Y(t) = (Y_0(t), Y_1(t), \dots)$ and $Z(t) = (Z_0(t), Z_1(t), \dots)$ are simple symmetric random walks and therefore recurrent, in that $Y_n(t) = 0$ for infinitely many values of n almost surely, and similarly for $Z_n(t)$. We have seen in Theorem 1.6 that recurrence for Y is *dynamically stable* in that

$$\mathbb{P}(\forall t \geq 0,\ Y_n(t) = 0 \text{ for infinitely many values of } n) = 1.$$

Our main result is that, in contrast, recurrence for Z is dynamically sensitive. Define

$$\mathcal{E} = \{t \in [0,1] : Z_n(t) \to \infty \text{ as } n \to \infty\},$$

$$\mathcal{E}_0 = \{t \in [0,1] : \liminf_{n\to\infty} Z_n(t) > 0\},$$

and more generally for $\alpha \geq 0$,

$$\mathcal{E}_\alpha = \left\{t \in [0,1] : \liminf_{n\to\infty} \frac{Z_n(t)}{n^\alpha} > 0\right\}.$$

Theorem 2.1. *There exist exceptional times of transience for the switch random walk: $\mathcal{E}$ is non-empty almost surely. In fact, the Hausdorff dimension of $\mathcal{E}_\alpha$ equals $1/2$ almost surely for any $\alpha \in [0, 1/2)$. On the other hand, $\mathcal{E}_\alpha$ is empty almost surely for any $\alpha > 1/2$.*

It is an interesting question as to whether $\mathcal{E}_{1/2}$ is empty or not. It is possible that the methods that we use to prove Theorem 2.1 could be extended to investigate this more delicate case, but this would require more detailed analysis of random walk sample paths that is beyond the scope of this work.

The case of $\alpha = 1$ corresponds to the Strong Law of Large Numbers being dynamically stable for Z, which matches Theorem 1.4 for Y.

Theorem 2.1 also implies that almost surely there are exceptional times for the Law of the Iterated Logarithm (LIL). That is,

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{Z_n(t)}{\sqrt{2n\log\log n}} = 1 \ \forall t \in [0,1]\right) = 0.$$

This is since Theorem 2.1 implies that there almost surely are times t where $Z(t)$ is negative for all large n so LIL must be dynamically sensitive for Z. This is in contrast with Theorem 1.5 for Y.

We also show that the event that Z_n is positive is noise sensitive. In fact we prove a stronger quantitative noise sensitivity result.

Theorem 2.2. *Let $(\varepsilon_n, n \geq 1)$ be any sequence in $(0,1)$ such that $n\varepsilon_n \to \infty$. The sequence of events $(\{Z_n > 0\}, n \geq 1)$ is quantitatively noise sensitive with respect to the sequence $(\varepsilon_n, n \geq 1)$, by which we mean that*

$$\mathbb{P}(Z_n(0) > 0 \text{ and } Z_n(\varepsilon_n) > 0) - \mathbb{P}(Z_n(0) > 0)^2 \to 0$$

as $n \to \infty$.

We note that the usual definition of (quantitative) noise sensitivity uses $-\log(1-\varepsilon_n)$ in place of ε_n above, but since $\varepsilon_n \in (0,1)$, this is equivalent to our statement.

We observe that if $\liminf n\varepsilon_n < \infty$, then for arbitrarily large values of n none of the first n bits are rerandomised by time ε_n, and therefore one cannot expect the events $\{Z_n(0) > 0\}$ and $\{Z_n(\varepsilon_n) > 0\}$ to decorrelate. In this sense Theorem 2.2 is as strong as it possibly could be; we say that the events $(\{Z_n > 0\}, n \geq 1)$ are *maximally noise sensitive.*

Again, Theorem 2.2 is in stark contrast to the corresponding statement for the compass random walk. In fact, the event that Y_n is positive is known to be noise *stable* [8], in that

$$\lim_{\varepsilon \to 0} \sup_n \mathbb{P}(\operatorname{sign} Y_n(0) \neq \operatorname{sign} Y_n(\varepsilon)) = 0.$$

We have seen this in our analysis of Example 3 in the previous chapter.

2.2 Structure and sketch proofs

2.2.1 Layout of chapter

This chapter is organised as follows. In Section 2.2.2 we outline some well-known facts about random walks that will be used extensively in our proofs. In Section 2.2.3 we give a rough sketch of the proofs of Theorems 2.1 and 2.2 that should give the reader an idea of the main arguments involved. We then carry out the proof of Theorem 2.2 in Section 2.3. The proof of Theorem 2.1 is substatially more complex, and we give an outline in Section 2.4, which reduces the bulk of the task to proving two propositions, Proposition 2.8 for the lower bound on the Hausdorff dimension and Proposition 2.12 for the upper bound, together with several technical lemmas. The proof of Proposition 2.8 is the most interesting part of the chapter and substantially different from existing proofs of related results. Rather than relying on the methods detailed in [19] such as randomised algorithms or the spectral sample, it instead uses more hands-on methods, leaning heavily on the independence of increments of random walks. We carry this out in Section 2.5. Then in Section 2.6 we prove Proposition 2.12, which mainly consists of elementary but intricate approximations. Finally, in Section 2.7 we prove the technical lemmas required to complete the proof of Theorem 2.1.

2.2.2 Notation and preparatory results

Throughout, we write $f(n) \lesssim g(n)$ if there exists a constant $c \in (0, \infty)$ such that $f(n) \leq cg(n)$ for all large n, and $f(n) \asymp g(n)$ if both $f(n) \lesssim g(n)$ and $g(n) \lesssim f(n)$. If there f and g are functions of multiple variables then we shall make it clear which variable is "large" and how the other variables relate to said variable. We use $\approx$ only in heuristics to mean "is roughly equal to". We write $\mathbb{P}_x$ for the probability measure under which our random walks begin from x, rather than 0. To be precise, we mean that under $\mathbb{P}_x$,

$$Z_n = x + \sum_{k=1}^{n} \prod_{j=1}^{k} X_j$$

and similarly for $Z_n(t)$, Y_n and $Y_n(t)$.

We will use the Fortuin-Kasteleyn-Ginibre (FKG) inequality [18] using the partial order on $\{-1, 1\}^{\mathbb{N}}$ given by setting $(x_1, x_2, \ldots) \leq (y_1, y_2, \ldots)$ if $x_i \leq y_i$ for all $i \in \mathbb{N}$. This says that if f and g are either both increasing functions or both decreasing functions with respect to this partial order, then

$$\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\mathbb{E}[g(X)] \tag{2.1}$$

and if f is increasing but g is decreasing, then

$$\mathbb{E}[f(X)g(X)] \leq \mathbb{E}[f(X)]\mathbb{E}[g(X)]. \tag{2.2}$$

We gather here some useful and well-known facts about simple symmetric random walks.

Lemma 2.3. *Uniformly over (j, z) such that $j \geq 2$ and $|z| \leq j^{3/4}$ and $z - j$ is even,* *we have that*

$$\mathbb{P}(Z_j = z) \asymp \frac{1}{j^{1/2}} \exp\Big(-\frac{z^2}{2j}\Big).$$

If $z-j$ is not even then $\mathbb{P}(Z_j = z) = 0$.

Proof. This is simply a version of the local central limit theorem: see for example [28, Proposition 2.5.3 and Corollary 2.5.4]. □

Lemma 2.4. *For any $j \geq 2$ and $x > 0$,*

$$\mathbb{P}(Z_j \geq x) \leq \exp\Big(-\frac{x^2}{2j}\Big).$$

Proof. This is an application of a simple Chernoff-style bound. For any $\lambda > 0$,

$$\mathbb{P}(Z_j \geq x) \leq \mathbb{E}[e^{\lambda Z_j}]e^{-\lambda x} = \mathbb{E}[e^{\lambda X_1}]^j e^{-\lambda x} = \Big(\frac{e^{\lambda}+e^{-\lambda}}{2}\Big)^j e^{-\lambda x}.$$

Noting that

$$\frac{e^{\lambda}+e^{-\lambda}}{2} = \sum_{i=0}^{\infty}\frac{\lambda^{2i}}{(2i)!} \leq \sum_{i=0}^{\infty}\frac{(\lambda^2/2)^i}{i!} = e^{\lambda^2/2},$$

we get

$$\mathbb{P}(Z_j \geq x) \leq \exp\Big(\frac{\lambda^2 j}{2} - \lambda x\Big)$$

and choosing $\lambda = x/j$ gives the result. □

Lemma 2.5. *For any $z, j \in \mathbb{N}$,*

$$\mathbb{P}(Z_i > -z \ \ \forall i = 1,\ldots,j) = \mathbb{P}(Z_j \in [-z+1, z]).$$

Proof. This is a version of the reflection principle. Note that

$$\begin{aligned}\mathbb{P}(Z_i > -z \ \ \forall i = 1,\ldots,j) &= \mathbb{P}(Z_i > -z \ \ \forall i = 1,\ldots,j,\ Z_j \geq -z+1)\\ &= \mathbb{P}(Z_j \geq -z+1) - \mathbb{P}(\exists i \leq j : Z_i \leq -z,\ Z_j \geq -z+1).\end{aligned}$$

Now by reflecting the random walk at the first hitting time of $-z$ (applying the strong Markov property), we have

$$\mathbb{P}(\exists i \leq j : Z_i \leq -z,\ Z_j \geq -z+1) = \mathbb{P}(Z_j \leq -z-1) = \mathbb{P}(Z_j \geq z+1),$$

which establishes the result. □

Corollary 2.6. *For any $n \geq 1$,*

$$\mathbb{P}(Z_i > 0 \ \ \forall i = 1, \dots, n) \asymp n^{-1/2}.$$

Proof. We have

$$\begin{aligned}\mathbb{P}(Z_i > 0 \ \ \forall i = 1, \dots, n) &= \mathbb{P}(Z_1 = 1,\, Z_i > 0 \ \ \forall i = 2, \dots, n) \\ &= \frac{1}{2}\mathbb{P}_1(Z_i > 0 \ \ \forall i = 1, \dots, n-1).\end{aligned}$$

Applying Lemma 2.5, the above equals $\frac{1}{2}\mathbb{P}_1(Z_{n-1} \in [0,1])$, and by Lemma 2.3 this is of order $n^{-1/2}$. □

2.2.3 Sketch proofs

For $t \geq 0$ let $I_0(t) = 0$, and for $k \geq 1$ define

$$I_k(t) = \min\{i > I_{k-1}(t) : X_i(t) \neq X_i(0)\},$$

the kth index for which our Rademacher random variables disagree at times 0 and t. We think of t being small, so that for many indices i we have $X_i(t) = X_i(0)$, and we call $I_k(t)$ the "kth change" (at time t relative to time 0). We call the steps of the random walk between $0 = I_0(t)$ and $I_1(t) - 1$ the *first period*, the steps between $I_1(t)$ and $I_2(t) - 1$ the *second period*, and so on. For each k we let $J_k(t) = I_k(t) - I_{k-1}(t)$ be the length of the kth period.

Our first key observation is that the increments of $Z_n(0)$ and $Z_n(t)$ are equal during odd periods (that is, for $n \in [I_{2k}, I_{2k+1}(t)-1]$); and the increments of $Z_n(0)$ and $-Z_n(t)$ are equal during even periods (that is, for $n \in [I_{2k+1}(t), I_{2k+2}(t) - 1]$). See Figure 2-1.

To see why Theorem 2.2 is true, let $t = \varepsilon \in (0,1)$ and run the random walks up to step n. Let $U_n(t)$ be the sum of the increments of $Z_n(0)$ over odd periods up to step n, and $V_n(t)$ be the sum of the increments over even periods up to step n. Then clearly

$$Z_n(0) = U_n(t) + V_n(t).$$

(Note that $U_n(t)$ and $V_n(t)$ depend on t because the periods depend on t, even though $Z_n(0)$ itself does not depend on t.) Of course, we can also write $Z_n(t)$ as the sum of its increments over odd periods, plus the sum of its increments over even periods. But the increments of $Z_n(t)$ over odd periods are equal to the increments of $Z_n(0)$ over odd periods, and the increments of $Z_n(t)$ over even periods are precisely *minus* the

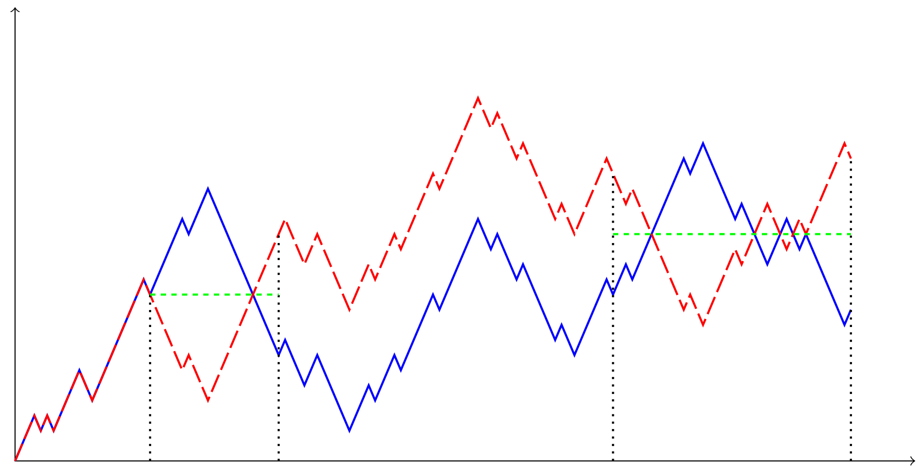

Figure 2-1: A realisation of $Z(0)$ in blue and $Z(t)$ in red (dashed) for the first four periods. The dotted green lines mark the lines of reflection.

increments of $Z_n(0)$ over even periods. Thus

$$Z_n(t) = U_n(t) - V_n(t).$$

We can think of $U_n(t)$ and $V_n(t)$ as being formed by cutting $Z_n(0)$ at the points $I_k(t)-1$ (for all k) and then rearranging and glueing such that all the steps shared between $Z_n(0)$ and $Z_n(t)$ occur first (so $U_n(t)$) and the steps that are reflected happen later ($V_n(t)$). See Figures 2-2, 2-3 and 2-4 to see this unfold.

As a result,

$$\begin{aligned}
\mathbb{P}(Z_n(0) > 0 \text{ and } Z_n(t) > 0) &= \mathbb{P}(U_n(t) + V_n(t) > 0 \text{ and } U_n(t) - V_n(t) > 0) \\
&= \mathbb{P}(U_n(t) > |V_n(t)|).
\end{aligned}$$

Now we note that—as long as $t \gg 1/n$, so that there are many periods by step n—the quantities $U_n(t)$ and $V_n(t)$ have *almost* the same distribution when n is large, and are *almost* independent. They are also symmetric and have small probability of being equal or equalling zero. If U and V are IID symmetric continuous random variables, then $\mathbb{P}(U > |V|) = 1/4$. Approximating this statement with $U_n(t)$ and $V_n(t)$ in place of U and V gives that

$$\mathbb{P}(Z_n(0) > 0 \text{ and } Z_n(t) > 0) \to 1/4$$

as $n \to \infty$, which is what is needed to prove Theorem 2.2 since clearly $\mathbb{P}(Z_n(0) > 0)^2 \to$

1/4.

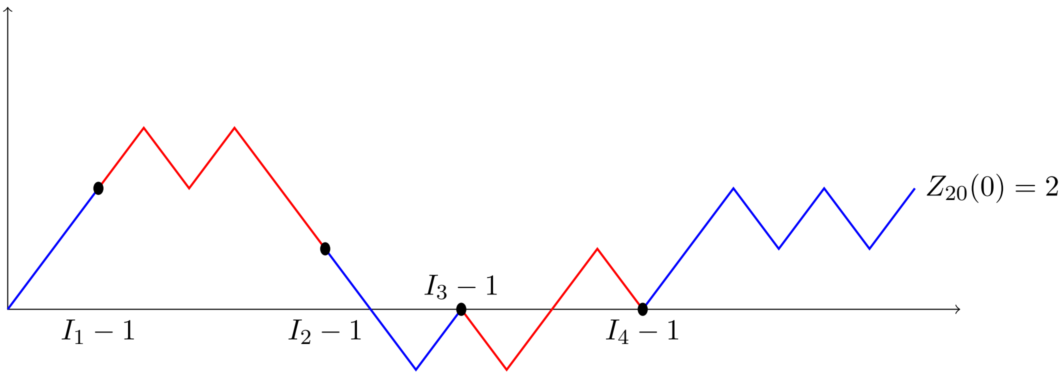

Figure 2-2: A realisation of the first 20 steps of $(Z_n(0))$, where in red are the segments affected by rerandomisations up until some time t

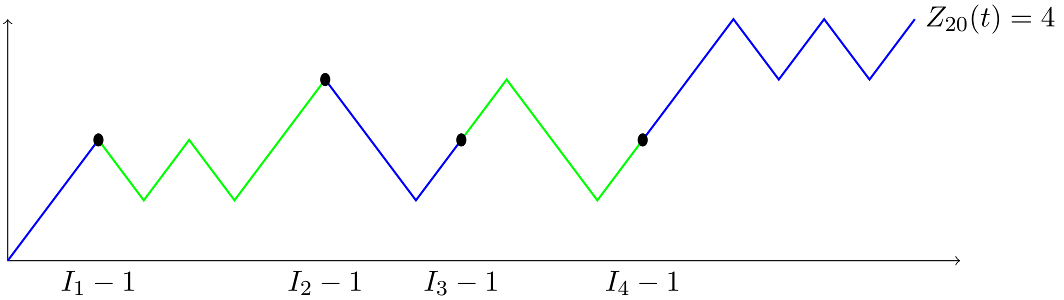

Figure 2-3: The same realisation as in Figure 2-2 but at time t. The green segments are reflections of the red segments in Figure 2-2.

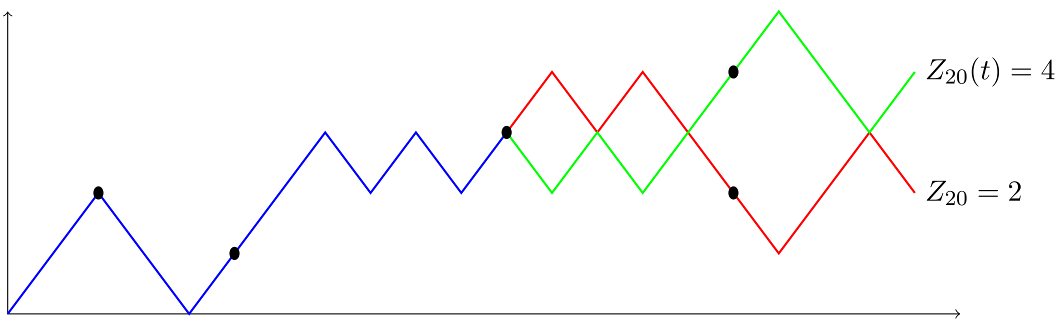

Figure 2-4: A combination of both Figures 2-2 and 2-3 where we have cut and glues the shared blue paths together, and stuck the reflected red/green paths together onto the end.

Theorem 2.1 is significantly more difficult to prove. We give a sketch of a proof of the existence of exceptional times, whose main ideas are also the key to the most difficult part of calculating the Hausdorff dimension of the set of such times. There will be a much more detailed proof outline in Section 2.4.

It is simpler to deal with $\mathcal{E}_0$ rather than $\mathcal{E}$ or $\mathcal{E}_\alpha$ for much of the proof. We define the event

$$P_n(t) = \{Z_k(t) > 0 \ \ \forall k \in \{1, \dots, n\}\},$$

that the random walk $Z(t)$ is positive for its first n steps, and consider

$$\kappa_n = \int_0^1 \mathbb{1}_{P_n(t)} \, \mathrm{d}t,$$

the Lebesgue amount of time in $[0, 1]$ during which $Z(t)$ stays positive for its first n steps. To show the existence of exceptional times, ignoring some technical issues, it essentially suffices to show that

$$\mathbb{E}[\kappa_n^2] \le C\mathbb{E}[\kappa_n]^2$$

for some finite constant C, from which we can deduce that $\mathbb{P}(\kappa_n > 0) \ge 1/C$ and let $n \to \infty$.

For the first moment, by Fubini's theorem and stationarity,

$$\mathbb{E}[\kappa_n] = \int_0^1 \mathbb{P}(P_n(t)) \, \mathrm{d}t = \int_0^1 \mathbb{P}(P_n(0)) \, \mathrm{d}t = \mathbb{P}(P_n(0)).$$

Corollary 2.6 tells us that $\mathbb{P}(P_n(0)) \asymp n^{-1/2}$.

For the second moment, again applying Fubini's theorem and stationarity, a simple argument (using Fubini's theorem and stationarity, and which we will give in full later) gives

$$\mathbb{E}[\kappa_n^2] \le 2 \int_0^1 \mathbb{P}(P_n(0) \cap P_n(t)) \, \mathrm{d}t.$$

Our task is therefore to show that $\int_0^1 \mathbb{P}(P_n(0) \cap P_n(t)) \, \mathrm{d}t \lesssim \mathbb{P}(P_n(0))^2 \asymp n^{-1}$.

During the even periods, the increments of $Z(0)$ and $Z(t)$ are mirrored. One can use this to show that the probability that both $Z(0)$ and $Z(t)$ remain positive over an even period is smaller than the square of the probability that $Z(0)$ stays positive over the same period. The total length of the even periods is roughly $n/2$ provided t is not too small, and so (skipping over several important details) we might hope that, at least when t is not too small,

$$\mathbb{P}(P_n(0) \cap P_n(t)) \lesssim \mathbb{P}(P_{n/2}(0))^2.$$

The details required to show this involve sewing together the increments over the even periods to create one random walk path of length roughly $n/2$. It is possible to do this in a very simple and natural way, except for one remaining issue: we cannot ignore the first period, on which the two random walks $Z(0)$ and $Z(t)$ are equal. On this period clearly the best upper bound we can get on the probability that both random walks stay positive is simply $\mathbb{P}(P_{I_1(t)-1}(0))$, rather than this quantity squared. A more reasonable overall upper bound is therefore

$$\mathbb{P}(P_n(0) \cap P_n(t)) \lesssim \frac{\mathbb{P}(P_{n/2}(0))^2}{\mathbb{P}(P_{I_1(t)-1}(0))}.$$

This does indeed hold, and since $I_1(t) \approx 2/t$, we have $\mathbb{P}(P_{I_1(t)-1}(0)) \asymp (2/t)^{-1/2}$, so that

$$\int_0^1 \mathbb{P}(P_n(0) \cap P_n(t))\,\mathrm{d}t \lesssim \int_0^1 \frac{n^{-1}}{t^{1/2}}\,\mathrm{d}t \asymp n^{-1}$$

as required. One may further note that an extra factor of $t^{-\gamma}$ in the integral would not make any difference to the calculation provided that $\gamma < 1/2$, which combined with Frostman's lemma essentially gives us the lower bound of $1/2$ on the Hausdorff dimension.

2.3 Proof of Theorem 2.2: noise sensitivity for $\{Z_n > 0\}$

Fix a sequence $(\varepsilon_n, n \geq 1)$ with $\varepsilon_n \in (0,1)$ for all n and $n\varepsilon_n \to \infty$. Many of the definitions in this section will depend implicitly on ε_n. Recall that for $t \geq 0$ we defined $I_0(t) = 0$, and for $k \geq 1$,

$$I_k(t) = \min\{i > I_{k-1}(t) : X_i(t) \neq X_i(0)\},$$

the start of the $(k+1)$th period. Let

$$K(n) = 2\lfloor n(1 - e^{-\varepsilon_n})/4 \rfloor.$$

We note that, since each X_i has rerandomised by time ε_n with probability $1 - e^{-\varepsilon_n}$, the period length $I_k(\varepsilon_n) - I_{k-1}(\varepsilon_n)$ is a Geometric random variable of parameter $(1 - e^{-\varepsilon_n})/2$. Thus by the law of large numbers we have $I_{K(n)}(\varepsilon_n) \approx n$.

There will be three main parts to this proof. In the first part, we show that the probability that the sum of the increments of a random walk on the odd periods is larger than the modulus of the sum of the increments on the even periods converges to $1/4$. In the second part, we will prove Theorem 2.2 but with $I_{K(n)}(\varepsilon_n)$ in place of n.

Finally, in the third part, we will transfer from using $I_{K(n)}(\varepsilon_n)$ to n.

Part 1: Probability that sum of increments on odd periods exceed modulus of sum of increments on even periods converges to $1/4$.
Define

$$U_n = \sum_{i=1}^{I_1(\varepsilon_n)-1} X_i + \sum_{i=I_2(\varepsilon_n)}^{I_3(\varepsilon_n)-1} X_i + \ldots + \sum_{i=I_{K(n)-2}(\varepsilon_n)}^{I_{K(n)-1}(\varepsilon_n)-1} X_i + X_{I_{K(n)}(\varepsilon_n)}$$

and

$$V_n = \sum_{i=I_1(\varepsilon_n)}^{I_2(\varepsilon_n)-1} X_i + \sum_{i=I_3(\varepsilon_n)}^{I_4(\varepsilon_n)-1} X_i + \ldots + \sum_{i=I_{K(n)-1}(\varepsilon_n)}^{I_{K(n)}(\varepsilon_n)-1} X_i.$$

In words, U_n is the sum of the increments of a simple symmetric random walk (in fact Y, though this is not important) over the odd periods up to step roughly n, and V_n is the sum over the even periods up to step roughly n. This is, of course, not quite true, since $I_{K(n)}(\varepsilon_n)$ is unlikely to be exactly n. On the positive side, this gives U_n and V_n some nice properties: in particular, they are identically distributed.

We claim that

$$\lim_{n\to\infty} \mathbb{P}(U_n + V_n > 0 \text{ and } U_n - V_n > 0) = \lim_{n\to\infty} \mathbb{P}(U_n > |V_n|) = 1/4.$$

To see this, we observe that

$$\begin{aligned}
1 = &\mathbb{P}(U_n > V_n > 0) + \mathbb{P}(U_n > -V_n > 0) + \mathbb{P}(V_n > U_n > 0) + \mathbb{P}(-V_n > U_n > 0) \\
&+ \mathbb{P}(U_n < V_n < 0) + \mathbb{P}(U_n < -V_n < 0) + \mathbb{P}(V_n < U_n < 0) + \mathbb{P}(-V_n < U_n < 0) \\
&\quad + \mathbb{P}(U_n = 0 \text{ or } V_n = 0 \text{ or } U_n = V_n \text{ or } U_n = -V_n).
\end{aligned}$$

The first eight terms are all equal, and the last tends to 0 as $n \to \infty$. Thus

$$\begin{aligned}
\mathbb{P}(U_n > |V_n|) &= \mathbb{P}(U_n > V_n > 0) + \mathbb{P}(U_n > -V_n > 0) + \mathbb{P}(U_n > V_n = 0) \\
&\to 1/8 + 1/8 + 0 = 1/4
\end{aligned}$$

as claimed.

Part 2: Proving Theorem 2.2 but with $I_{K(n)}(\varepsilon_n)$ in place of n.

Noting that $K(n)$ is even, we now let

$$U'_n = Z_{I_1(\varepsilon_n)-1}(0)+\sum_{\substack{k=3\\ k \text{ odd}}}^{K(n)-1} \big(Z_{I_k(\varepsilon_n)-1}(0)-Z_{I_{k-1}(\varepsilon_n)-1}(0)\big)+Z_{I_{K(n)}(\varepsilon_n)}(0)-Z_{I_{K(n)}(\varepsilon_n)-1}(0)$$

and

$$V'_n = \sum_{\substack{k=2\\ k \text{ even}}}^{K(n)} (Z_{I_k(\varepsilon_n)-1}(0) - Z_{I_{k-1}(\varepsilon_n)-1}(0)).$$

Clearly we have $Z_{I_{K(n)}(\varepsilon_n)}(0) = U'_n + V'_n$. Moreover, since the increments of $Z(\varepsilon_n)$ and $Z(0)$ are equal on odd periods and mirrored on even periods, we have

$$Z_{I_{K(n)}(\varepsilon_n)}(\varepsilon_n) = U'_n - V'_n.$$

Thirdly, note that (again recalling that $K(n)$ is even) U'_n and V'_n have the same joint distribution as U_n and V_n. Thus we have

$$\begin{aligned}\mathbb{P}(Z_{I_{K(n)}(\varepsilon_n)}(0) > 0 \text{ and } Z_{I_{K(n)}(\varepsilon_n)}(\varepsilon_n) > 0) &= \mathbb{P}(U'_n + V'_n > 0 \text{ and } U'_n - V'_n > 0)\\ &= \mathbb{P}(U_n + V_n > 0 \text{ and } U_n - V_n > 0)\end{aligned}$$

which we have just shown (in Part 1) converges to $1/4$ as $n \to \infty$. Thus

$$\mathbb{P}(Z_{I_{K(n)}(\varepsilon_n)}(0) > 0 \text{ and } Z_{I_{K(n)}(\varepsilon_n)}(\varepsilon_n) > 0) - \mathbb{P}(Z_{I_{K(n)}(\varepsilon_n)}(0) > 0)^2 \to \frac{1}{4} - \Big(\frac{1}{2}\Big)^2 = 0,$$

establishing the theorem with $I_{K(n)}(\varepsilon_n)$ in place of n.

We remark here that so far, the proof works for any value of $\varepsilon_n \in (0,1)$. However, if ε_n is too small, then the value of $K(n)$ is not large, which will cause problems in the following.

Part 3: Transferring from $I_{K(n)}(\varepsilon_n)$ to n.
We claim that

$$\mathbb{P}(Z_n(0) > 0 \text{ and } Z_n(\varepsilon_n) > 0) = \mathbb{P}(Z_{I_{K(n)}(\varepsilon_n)}(0) > 0 \text{ and } Z_{I_{K(n)}(\varepsilon_n)}(\varepsilon_n) > 0) + o(1). \tag{2.3}$$

We will use the elementary bounds, for any events A, B, A' and B',

$$\mathbb{P}(A \cap B) \leq \mathbb{P}(A' \cap B') + \mathbb{P}(A \setminus A') + \mathbb{P}(B \setminus B')$$

and

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A' \cap B') - \mathbb{P}(A' \setminus A) - \mathbb{P}(B' \setminus B).$$

For the upper bound, using the first fact above,

$$\begin{aligned}\mathbb{P}(Z_n(0) > 0 \text{ and } Z_n(\varepsilon_n) > 0) \leq \mathbb{P}(Z_{I_{K(n)}(\varepsilon_n)}(0) > 0 \text{ and } Z_{I_{K(n)}(\varepsilon_n)}(\varepsilon_n) > 0)&\\ + \mathbb{P}(Z_n(0) > 0 \text{ but } Z_{I_{K(n)}(\varepsilon_n)}(0) \leq 0)&\\ + \mathbb{P}(Z_n(\varepsilon_n) > 0 \text{ but } Z_{I_{K(n)}(\varepsilon_n)}(\varepsilon_n) \leq 0)&,\end{aligned}$$

and for the lower bound, using the second fact above,

$$\begin{aligned}\mathbb{P}(Z_n(0) > 0 \text{ and } Z_n(\varepsilon_n) > 0) \geq \mathbb{P}(Z_{I_{K(n)}(\varepsilon_n)}(0) > 0 \text{ and } Z_{I_{K(n)}(\varepsilon_n)}(\varepsilon_n) > 0)&\\ - \mathbb{P}(Z_n(0) \leq 0 \text{ but } Z_{I_{K(n)}(\varepsilon_n)}(0) > 0)&\\ - \mathbb{P}(Z_n(\varepsilon_n) \leq 0 \text{ but } Z_{I_{K(n)}(\varepsilon_n)}(\varepsilon_n) > 0)&.\end{aligned}$$

We will show that

$$\mathbb{P}(Z_n(0) > 0 \text{ but } Z_{I_{K(n)}(\varepsilon_n)}(0) \leq 0) \to 0;$$

the three other similar terms can be dealt with similarly. To do this, we first note that for any $x_n, y_n > 0$,

$$\begin{aligned}\mathbb{P}\big(Z_n(0) > 0 \text{ but } Z_{I_{K(n)}(\varepsilon_n)}(0) \leq 0\big) \leq \mathbb{P}\big(|I_{K(n)}(\varepsilon_n) - n| > x_n\big) + \mathbb{P}\big(Z_n(0) \in (0, y_n)\big)&\\ + \mathbb{P}\Big(Z_n(0) \geq y_n \text{ but } \min_{j\in[n-x_n,n+x_n]} Z_j(0) \leq 0\Big)&. \quad (2.4)\end{aligned}$$

We first consider $\mathbb{P}(|I_{K(n)}(\varepsilon_n) - n| > x_n)$. We use Markov's inequality to see that

$$\mathbb{P}\big(|I_{K(n)}(\varepsilon_n) - n| > x_n\big) \leq \frac{\mathbb{E}\big[|I_{K(n)}(\varepsilon_n) - n|^2\big]}{x_n^2},$$

and using the fact that $I_{K(n)}(\varepsilon_n)$ is a sum of $K(n)$ independent Geometric random variables of parameter $(1 - e^{-\varepsilon_n})/2$, we have

$$\begin{aligned}\mathbb{E}\big[|I_{K(n)}(\varepsilon_n) - n|^2\big] &= \mathrm{Var}(I_{K(n)}(\varepsilon_n)) + \mathbb{E}[I_{K(n)}(\varepsilon_n)]^2 - 2n\mathbb{E}[I_{K(n)}(\varepsilon_n)] + n^2\\ &= \frac{2K(n)(1 + e^{-\varepsilon_n})}{(1 - e^{-\varepsilon_n})^2} + \frac{4K(n)^2}{(1 - e^{-\varepsilon_n})^2} - \frac{4nK(n)}{1 - e^{-\varepsilon_n}} + n^2.\end{aligned}$$

Recall that $K(n) = 2\lfloor n(1 - e^{-\varepsilon_n})/4 \rfloor$. The above expression is at most

$$\frac{n(1 + e^{-\varepsilon_n})}{1 - e^{-\varepsilon_n}} + n^2 - \Big(\frac{8n}{1 - e^{-\varepsilon_n}}\Big)\Big(\frac{n(1 - e^{-\varepsilon_n})}{4} - 1\Big) + n^2 \leq \frac{10n}{1 - e^{-\varepsilon_n}}.$$

Thus

$$\mathbb{P}\big(|I_{K(n)}(\varepsilon_n) - n| > x_n\big) \le \frac{10n}{x_n^2(1-e^{-\varepsilon_n})}.$$

Choosing the value $x_n = n^{5/8}/(1-e^{-\varepsilon_n})^{3/8}$, we have

$$\mathbb{P}\big(|I_{K(n)}(\varepsilon_n) - n| > x_n\big) \le \frac{10}{n^{1/4}(1-e^{-\varepsilon_n})^{1/4}} \to 0 \tag{2.5}$$

by our assumption that $n\varepsilon_n \to \infty$. We now move on to the second term on the right-hand side of (2.4). Choosing $y_n = n^{3/8}/\varepsilon_n^{1/8}$, since $(Z_j(0), j \ge 0)$ is a simple symmetric random walk and $y_n \ll n^{1/2}$, by the central limit theorem we have

$$\mathbb{P}\big(Z_n(0) \in (0, y_n)\big) \to 0. \tag{2.6}$$

For the final term in (2.4), we split according to if $(Z_j(0))$ is first non-positive in $[n - x_n, n + x_n]$ before or after step n. In the former case the walk must grow by at least y_n in at most x_n steps, while in the latter case the walk must fall by at least y_n in at most x_n steps. With this in mind, the strong Markov property and Lemma 2.5 gives,

$$\begin{aligned}\mathbb{P}\Big(Z_n(0) \ge y_n \text{ but } \min_{j\in[n-x_n,n+x_n]} Z_j(0) \le 0\Big) &\le \mathbb{P}_0\Big(\max_{j\in[0,x_n]} Z_j(0) \ge y_n\Big) \\ &\quad + \mathbb{P}_{y_n}\Big(\min_{j\in[0,x_n]} Z_j(0) \le 0\Big) \\ &= 2\big(1 - \mathbb{P}(Z_{\lfloor x_n\rfloor}(0) \in [-y_n+1, y_n])\big).\end{aligned}$$

Since $x_n = n^{5/8}/(1-e^{-\varepsilon_n})^{3/8} \ll n^{6/8}/\varepsilon_n^{2/8} = y_n^2$, the central limit theorem tells us that the above also converges to zero as $n \to \infty$. Combining this with (2.5) and (2.6), we see from (2.4) that

$$\mathbb{P}\big(Z_n(0) > 0 \text{ but } Z_{I_{K(n)}(\varepsilon_n)}(0) \le 0\big) \to 0.$$

This, together with very similar bounds on the other three terms mentioned above, establishes (2.3). In Part 2 we showed that

$$\lim_{n\to\infty} \mathbb{P}(Z_{I_{K(n)}(\varepsilon_n)}(0) > 0 \text{ and } Z_{I_{K(n)}(\varepsilon_n)}(\varepsilon_n) > 0) = 1/4,$$

and clearly $\mathbb{P}(Z_n(0) > 0) \to 1/2$, so the proof of Theorem 2.2 is complete.

2.4 Outline of the proof of Theorem 2.1: Hausdorff dimension of exceptional times is $1/2$

We now outline the main steps in turning the heuristic in Section 2.2.3 into a rigorous proof that the Hausdorff dimension of

$$\mathcal{E}_\alpha = \Big\{t \in [0,1] : \liminf_{n\to\infty} \frac{Z_n(t)}{n^\alpha} > 0\Big\}$$

is 1/2 almost surely for any $\alpha \in [0, 1/2)$. Since $\mathcal{E}_\alpha \subset \mathcal{E}_0$ for any $\alpha \geq 0$, it suffices to give an upper bound on the dimension of $\mathcal{E}_0$ and a lower bound on the dimension of $\mathcal{E}_\alpha$ for $\alpha \in (0, 1/2)$. This also, of course, implies that $\mathcal{E}$ is non-empty almost surely and therefore that there exist exceptional times of transience. We will proceed by stating a series of results, whose proofs we delay until later sections.

2.4.1 Lower bound on Hausdorff dimension of $\mathcal{E}_\alpha$

We start this subsection in further generality than is needed, as we need this material as well in Chapter 3.

Let (A_n) be a sequence of events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Build a larger probability space with dynamical time so that $A_n(t)$, the event that A_n occurs at time t, is well defined. Let

$$T_n = \{t \in [0,1] \ : \ A_n(t) \text{ holds}\}$$

and $T = \cap_n T_n$. Also define $\bar{T}_n$ as the closure of T_n. Now define for $\gamma \in [0, 1)$,

$$\Phi_n(\gamma) = \frac{1}{\mathbb{P}(A_n)^2} \int_0^1 \int_0^1 \frac{\mathbb{1}_{A_n(s)\cap A_n(t)}}{|t-s|^\gamma} \,\mathrm{d}s\,\mathrm{d}t.$$

We have the following result for producing a lower bound on the Hausdorff dimension we want. This result is based on the following corollary of [44, Lemma 6.2], which in turn is an application of Frostman's lemma. We shall prove the following result in Section 2.7.

Lemma 2.7. *Suppose that for some $\alpha \geq 0$ and $\gamma \in (0, 1)$ we have*

$$\sup_n \mathbb{E}[\Phi_n(\gamma)] < \infty.$$

Then the Hausdorff dimension of $\bigcap_n \bar{T}_n$ is at least γ with strictly positive probability.

Now we define the objects we are going to use in this chapter. As in the sketch proof, we define the event

$$P_n(t) = \{Z_i(t) > 0 \ \ \forall i = 1, \ldots, n\}$$

that the random walk $Z(t)$ is positive up to step n, and similarly

$$P_n = \{Z_i > 0 \ \ \forall i = 1, \ldots, n\}.$$

We will use these events for much of the proof. However, to consider $\mathcal{E}_\alpha$ for $\alpha > 0$, we will also need the more complicated events

$$P_n^\alpha(t) = \big\{Z_i(t) \geq i^\alpha \ \ \forall i = 1, \ldots, n\big\}$$

that the random walk $Z(t)$ remains above the curve i^α for all steps $i \leq n$, and similarly for P_n^α. Here we could consider any $\alpha \geq 0$, though we will mostly think of $\alpha \in [0, 1/2)$. Note that $P_n^0(t) = P_n(t)$. Let

$$T_n^\alpha = \{t \in [0,1] : P_n^\alpha(t) \text{ holds}\}$$

be the set of times at which the random walk stays above the curve i^α up to step n. We write $\bar{T}_n^\alpha$ for the closure of T_n^α and $T^\alpha = \bigcap_n T_n^\alpha$. Finally define, for $\gamma \in [0, 1)$,

$$\Phi_n^\alpha(\gamma) = \frac{1}{\mathbb{P}(P_n^\alpha)^2} \int_0^1 \int_0^1 \frac{\mathbb{1}_{P_n^\alpha(s) \cap P_n^\alpha(t)}}{|t-s|^\gamma} \, \mathrm{d}s \, \mathrm{d}t.$$

Given Lemma 2.7, our main task in proving the lower bound we require becomes to show that $\mathbb{E}[\Phi_n^\alpha(\gamma)]$ is bounded above for each $\alpha, \gamma < 1/2$. This will be the most difficult (and most novel) part of our proof, and will be carried out in Section 2.5.

Proposition 2.8. *For any $\alpha, \gamma \in [0, 1/2)$,*

$$\sup_n \mathbb{E}[\Phi_n^\alpha(\gamma)] < \infty.$$

Combining Lemma 2.7 and Proposition 2.8 tells us that for any $\alpha, \gamma \in [0, 1/2)$, the Hausdorff dimension of $\bigcap_n \bar{T}_n^\alpha$ is at least γ with strictly positive probability. This is not quite what was promised in Theorem 2.1, which in fact says that the Hausdorff dimension of $\mathcal{E}_\alpha$ is $1/2$ almost surely for any $\alpha \in [0, 1/2)$. Moving from $\bigcap_n \bar{T}_n^\alpha$ to T^α is a technicality that can be handled in basically the same way as [36, Lemma 3.2]; and of course $T^\alpha \subset \mathcal{E}_\alpha$. Finally, showing that the Hausdorff dimension of $\mathcal{E}_\alpha$ is at least $1/2$ almost surely, rather than with positive probability, follows from standard ergodicity arguments (of course this cannot hold for T^α, since with positive probability $Z_2(t) = 0$

for all $t \in [0,1]$). The following lemmas take care of these steps. We will prove them in Section 2.7.

Lemma 2.9. *For any $\alpha \geq 0$, we have*

$$\bigcap_{n=1}^{\infty} \bar{T}_n^{\alpha} = \bigcap_{n=1}^{\infty} T_n^{\alpha}$$

almost surely.

Lemma 2.10. *For each $\alpha \geq 0$, the Hausdorff dimension of $\mathcal{E}_\alpha$ is a constant (possibly depending on α) almost surely.*

2.4.2 Upper bound on Hausdorff dimension of $\mathcal{E}_0$

The following definitions are more or less standard in the noise sensitivity literature. For a function $f : \{-1,1\}^{\mathbb{N}} \to \mathbb{R}$ and random variables $X_1, X_2, \ldots$ taking values in $\{-1,1\}$, we say that $m \in \mathbb{N}$ is *pivotal* for f if

$$f(X_1, \ldots, X_{m-1}, X_m, X_{m+1}, X_{m+2}, \ldots) \neq f(X_1, \ldots, X_{m-1}, -X_m, X_{m+1}, X_{m+2}, \ldots).$$

Of course this definition depends on the realisation of $X_1, X_2, \ldots$, although we note that it is independent of the value of $X_m \in \{-1,1\}$. For an event E, we say that m is pivotal for E if m is pivotal for the indicator function of E. We define the *influence* of the mth bit (on E) to be

$$\mathcal{I}_m(E) = \mathbb{P}(m \text{ is pivotal for } E)$$

and the *total influence* of E to be

$$\mathcal{I}(E) = \sum_{m=1}^{\infty} \mathcal{I}_m(E).$$

For technical reasons, we will need the following generalisations of P_n and T. For $k \in 2\mathbb{Z}_+$, define the event

$$P_{k,n} = \{Z_k = 0,\ Z_i > 0\ \forall i = k+1, \ldots, k+n\}$$

that Z is zero at step k and positive for the next n steps, and let

$$T'_k = \{t \in [0,1] : Z_k(t) = 0,\ Z_i(t) > 0 \ \ \forall i = k+1, k+2, \ldots\}$$

be the set of times at which $Z_k(t)$ is zero and $Z_i(t)$ is strictly positive from step $k+1$ onwards.

Our next lemma is just a rephrasing of [44, Theorem 8.1] into our setting, and gives us a condition for bounding the Hausdorff dimension of T_k' in terms of the total influence of $P_{k,n}$.

Lemma 2.11. *The Hausdorff dimension of T_k' is almost surely at most*

$$\liminf_{n\to\infty}\Big(1-\frac{\log\mathbb{P}(P_{k,n})}{\log\mathcal{I}(P_{k,n})}\Big)^{-1}.$$

Proof. This is almost exactly the second part of the statement of [44, Theorem 8.1] translated into our notation. There is an extra condition that the events $P_{k,n}$ must depend only on finitely many random variables, but this is clearly satisfied since $P_{k,n}$ depends only on $X_1,\dots,X_{n+k}$.

It is also necessary that $\mathcal{I}(P_{k,n})\to\infty$ as $n\to\infty$ (for fixed k), but this is clear given Proposition 2.12 below alongside the fact that $\mathcal{I}_m(P_{k,n})=\mathbb{P}(Z_k=0)\mathcal{I}_{m-k}(P_n)$ whenever $m>k$, as then

$$\begin{aligned}\mathcal{I}(P_{k,n}) &= \sum_{m=1}^{k}\mathcal{I}_m(P_{k,n})+\sum_{m=k+1}^{k+n}\mathcal{I}_m(P_{k,n})\\ &\lesssim k+\frac{\mathbb{P}(Z_k=0)}{n^{3/2}}\sum_{m=1}^{n}(n-m+1)\asymp k+\mathbb{P}(Z_k=0)n^{1/2}.\end{aligned}$$

(the above display and facts preceding it are fully justified in Section 2.4.4). □

To implement Lemma 2.11 we now need an upper bound on the influences of P_n.

Proposition 2.12. *Uniformly for $1\le m\le n$, we have as $n\to\infty$ that*

$$\mathcal{I}_m(P_n)\asymp\frac{n-m+1}{n^{3/2}}.$$

This result will be proved in Section 2.6. Combining Proposition 2.12 with Lemma 2.11 will give us the upper bound of 1/2 on the Hausdorff dimension of T^0 and hence $\mathcal{E}$. We carry out the details in Section 2.4.4.

2.4.3 $\mathcal{E}_\alpha$ is empty for $\alpha > 1/2$

The final part of Theorem 2.1 says that $\mathcal{E}_\alpha$ is empty almost surely when $\alpha > 1/2$. The proof of this fact follows a fairly standard argument. For $\alpha, t \geq 0$ and $n \in \mathbb{N}$ define the event $L_n^\alpha(t) = \{Z_n(t) \geq n^\alpha\}$, and for $k \in \mathbb{N}$ let $\mathcal{L}_n^\alpha(k) = \int_0^k \mathbb{1}_{L_n^\alpha(t)}\,\mathrm{d}t$. Note that

$$\mathbb{P}(\mathcal{L}_n^\alpha(1) > 0) \leq \mathbb{P}(\mathcal{L}_n^\alpha(1) > 0)\frac{\mathbb{E}[\mathcal{L}_n^\alpha(2)]}{\mathbb{E}[\mathcal{L}_n^\alpha(2)\mathbb{1}_{\{\mathcal{L}_n^\alpha(1)>0\}}]} = \frac{\mathbb{E}[\mathcal{L}_n^\alpha(2)]}{\mathbb{E}[\mathcal{L}_n^\alpha(2)\,|\,\mathcal{L}_n^\alpha(1) > 0]}. \tag{2.7}$$

By Fubini's theorem and stationarity,

$$\mathbb{E}[\mathcal{L}_n^\alpha(2)] = \int_0^2 \mathbb{P}(Z_n(t) \geq n^\alpha)\,\mathrm{d}t = 2\mathbb{P}(Z_n \geq n^\alpha).$$

By Markov's inequality, for any $\lambda > 0$,

$$\mathbb{P}(Z_n \geq n^\alpha) = \mathbb{P}(\exp(\lambda Z_n) \geq \exp(\lambda n^\alpha)) \leq \mathbb{E}[\exp(\lambda Z_n)]\exp(-\lambda n^\alpha).$$

Since Z_n is a sum of n independent and identically distributed random variables,

$$\mathbb{E}[\exp(\lambda Z_n)] = \mathbb{E}[\exp(\lambda Z_1)]^n = (e^\lambda/2 + e^{-\lambda}/2)^n.$$

When $\lambda \leq 1$ we have $e^\lambda/2 + e^{-\lambda}/2 \leq 1 + 3\lambda^2/4$, so fixing $\alpha \in (1/2, 1]$ and choosing $\lambda = n^{\alpha-1}$, we have

$$\mathbb{E}[\exp(\lambda Z_n)] \leq \Big(1 + \frac{3}{4}\lambda^2\Big)^n = \Big(1 + \frac{3}{4}n^{2\alpha-2}\Big)^n \leq \exp\Big(\frac{3}{4}n^{2\alpha-1}\Big).$$

Thus, again with $\alpha \in (1/2, 1]$ and $\lambda = n^{\alpha-1}$,

$$\mathbb{E}[\mathcal{L}_n^\alpha(2)] = 2\mathbb{P}(Z_n \geq n^\alpha) \leq 2\exp\Big(\frac{3}{4}n^{2\alpha-1}\Big)\exp(-n^{2\alpha-1}) = 2\exp(-n^{2\alpha-1}/4). \tag{2.8}$$

On the other hand, letting $T = \inf\{t \in [0,1] : Z_n(t) \geq n^\alpha\}$, we have

$$\mathbb{E}[\mathcal{L}_n^\alpha(2)\,|\,\mathcal{L}_n^\alpha(1) > 0] \geq \mathbb{E}\Big[\int_T^{T+1} \mathbb{1}_{L_n^\alpha(t)}\,\mathrm{d}t\,\Big|\,\mathcal{L}_n^\alpha(1) > 0\Big].$$

Let $T' = \inf\{t \geq T : \text{one of the first } n \text{ steps rerandomises}\}$. Then clearly, provided $T < \infty$,

$$\int_T^{T+1} \mathbb{1}_{L_n^\alpha(t)}\,\mathrm{d}t \geq (T' - T) \wedge 1.$$

However, by the strong Markov property, $T' - T$ is exponentially distributed with parameter n. Thus

$$\mathbb{E}\Big[\int_T^{T+1} \mathbb{1}_{L_n^\alpha(t)}\,\mathrm{d}t \,\Big|\, \mathcal{F}_T\Big] \geq \mathbb{E}[(T'-T)\wedge 1] = \int_0^1 s\cdot ne^{-ns}\,\mathrm{d}s \geq \int_0^{1/n} nse^{-ns}\,\mathrm{d}s \geq \frac{1}{2en},$$

and therefore

$$\mathbb{E}[\mathcal{L}_n^\alpha(2)\,|\,\mathcal{L}_n^\alpha(1)>0] \geq \frac{1}{2en}.$$

Combining this with (2.7) and (2.8), for any $\alpha \in (1/2,1]$ we have

$$\mathbb{P}(\mathcal{L}_n^\alpha(1)>0) \leq 2\exp(-n^{2\alpha-1}/4)\cdot 2en.$$

For $\mathcal{E}_\alpha$ to be non-empty we must have a time $t\in[0,1]$ where there exists $N\in\mathbb{N}$ such that $L_n(t)$ occurs for all integer $n \geq N$. But when $\alpha\in(1/2,1]$, the Borel-Cantelli lemma gives that there exists $N\in\mathbb{N}$ such that $\mathcal{L}_n^\alpha = 0$ almost surely for all $n\geq N$. We now show that $\mathcal{L}_n^\alpha = 0$ implies $L_n^\alpha(t)$ does not occur for all $t\in[0,1]$.

If $L_n^\alpha(t)$ occurs for some $t\in[0,1)$ then this would force an interval of time of positive Lebesgue measure to exist where L_n^α occurs, as we would have to wait for the next rerandomisation time. This is clearly contradictory. We are then left to check that $L_n^\alpha(1)$ cannot occur infinitely often. By equality in law and Fubini's theorem, it can be seen that

$$\mathbb{P}(L_n^\alpha(1)) = \mathbb{P}(L_n^\alpha(0)) = \mathbb{E}[\mathcal{L}_n^\alpha(1)] \leq \frac{1}{2}\times(2.8) = \exp\left(-n^{2\alpha-1}/4\right).$$

Again, Borel-Cantelli shows that we cannot have $L_n^\alpha(1)$ occuring infinitely often, concluding the proof. Our result is trivially true for $\alpha>1$ as the maximum of a random walk after n steps is n.

2.4.4 Completing the proof of Theorem 2.1

We now tie together the results from Sections 2.4.1, 2.4.2 and 2.4.3 to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. We showed in Section 2.4.3 that $\mathcal{E}_\alpha$ is empty almost surely for $\alpha > 1/2$, so it remains to show that the Hausdorff dimension of $\mathcal{E}_\alpha$ is $1/2$ for any $\alpha\in[0,1/2)$. As stated at the beginning of Section 2.4, it suffices to show that the Hausdorff dimension of $\mathcal{E}_\alpha$ is at least $1/2$ for $\alpha>0$ and the Hausdorff dimension of $\mathcal{E}_0$ is at most $1/2$.

By Lemma 2.7 and Proposition 2.8, we know that for any $\alpha,\gamma\in[0,1/2)$, the Hausdorff dimension of $\bigcap_n \bar{T}_n^\alpha$ is at least γ with strictly positive probability. By Lemma 2.9, the same holds for T^α, and since $T^\alpha \subset \mathcal{E}_\alpha$, the same holds for $\mathcal{E}_\alpha$. Lemma 2.10 then tells us that the Hausdorff dimension of $\mathcal{E}_\alpha$ must be at least $1/2$ almost surely.

Moving on to the upper bound, take $k \in 2\mathbb{Z}_+$ and $m \in \{k+1, k+2, \ldots, k+n\}$. If $Z_k \neq 0$ then m cannot be pivotal for $P_{k,n}$, so

$$\begin{aligned}\mathcal{I}_m(P_{k,n}) &= \mathbb{P}(Z_k = 0,\ m \text{ is pivotal for } P_{k,n}) \\ &= \mathbb{P}(Z_k = 0)\mathbb{P}(m \text{ is pivotal for } P_{k,n} \,|\, Z_k = 0).\end{aligned}$$

But by the Markov property,

$$\mathbb{P}(m \text{ is pivotal for } P_{k,n} \,|\, Z_k = 0) = \mathbb{P}(m - k \text{ is pivotal for } P_n) = \mathcal{I}_{m-k}(P_n).$$

Thus

$$\mathcal{I}(P_{k,n}) = \sum_{m=1}^{k} \mathcal{I}_m(P_{k,n}) + \sum_{m=k+1}^{k+n} \mathcal{I}_m(P_{k,n}) \leq k + \mathbb{P}(Z_k = 0) \sum_{m=1}^{n} \mathcal{I}_m(P_n),$$

and so, applying Proposition 2.12, we have for large n and any fixed k that

$$\mathcal{I}(P_{k,n}) \lesssim k + \frac{\mathbb{P}(Z_k = 0)}{n^{3/2}} \sum_{m=1}^{n} (n - m + 1) \asymp k + \mathbb{P}(Z_k = 0) n^{1/2}. \tag{2.9}$$

By the Markov property

$$\mathbb{P}(P_{k,n}) = \mathbb{P}(Z_k = 0)\mathbb{P}(Z_i > 0 \ \forall i = k+1, k+2, \ldots, k+n \,|\, Z_k = 0) = \mathbb{P}(Z_k = 0)\mathbb{P}(P_n),$$

and by Corollary 2.6 we have $\mathbb{P}(P_n) \asymp n^{-1/2}$. Combining this with (2.9), we see that there exist constants $c, c' \in (0, \infty)$ such that

$$\frac{-\log \mathbb{P}(P_{k,n})}{\log \mathcal{I}(P_{k,n})} \geq \frac{\frac{1}{2}\log n - \log c - \log \mathbb{P}(Z_k = 0)}{\frac{1}{2}\log n + \log c' + \log(\mathbb{P}(Z_k = 0) + kn^{-1/2})},$$

which converges to 1 as $n \to \infty$ for each fixed k. From Lemma 2.11 we obtain that the Hausdorff dimension of T'_k is almost surely at most $(1+1)^{-1} = 1/2$.

Finally,

$$\mathcal{E}_0 = \{t \in [0,1] : \liminf_{n\to\infty} Z_n(t) > 0\} = \bigcup_k T'_k$$

which as a countable union of sets of Hausdorff dimension at most $1/2$ almost surely, itself has Hausdorff dimension at most 1/2 almost surely. This completes the proof. □

2.5 Proof of Proposition 2.8: bounding $\mathbb{E}[\Phi_n^\alpha(\gamma)]$ from above

First note that, by Fubini's theorem,

$$\mathbb{E}[\Phi_n^\alpha(\gamma)] = \frac{1}{\mathbb{P}(P_n^\alpha)^2}\mathbb{E}\Big[\int_0^1\int_0^1 \frac{\mathbb{1}_{P_n^\alpha(s)\cap P_n^\alpha(t)}}{|t-s|^\gamma}\,\mathrm{d}s\,\mathrm{d}t\Big]$$
$$= \frac{1}{\mathbb{P}(P_n^\alpha)^2}\int_0^1\int_0^1 \frac{\mathbb{P}(P_n^\alpha(s)\cap P_n^\alpha(t))}{|t-s|^\gamma}\,\mathrm{d}s\,\mathrm{d}t.$$

By stationarity, this is bounded above by

$$\frac{2}{\mathbb{P}(P_n^\alpha)^2}\int_0^1 \frac{\mathbb{P}(P_n^\alpha(0)\cap P_n^\alpha(t))}{t^\gamma}\,\mathrm{d}t,$$

and since $P_n^\alpha(u)\subset P_n(u)$ for any $\alpha, u\geq 0$, this is at most

$$\frac{2}{\mathbb{P}(P_n^\alpha)^2}\int_0^1 \frac{\mathbb{P}(P_n(0)\cap P_n(t))}{t^\gamma}\,\mathrm{d}t.$$

The following lemma says that the probability of P_n^α is of the same order as the probability as P_n. It is a simple application of [42, Theorem 2] and we will prove it later in this section.

Lemma 2.13. *For any $\alpha < 1/2$,*

$$\mathbb{P}(P_n^\alpha) \asymp \frac{1}{\sqrt{n}}.$$

We now want to bound $\mathbb{P}(P_n(0)\cap P_n(t))$. As suggested in the sketch proof in Section 2.2.3, the main idea is that on even periods two mirrored random walks (representing the walk at time 0 and time t) must both be larger than 0. The difficulty is in handling the dependencies between periods, and for this we need some more definitions. We recall first that $I_0(t) = 0$ and for $j\geq 1$

$$I_j(t) = \min\{i > I_{j-1}(t) : X_i(t) \neq X_i(0)\},$$

the jth index for which our Rademacher random variables disagree at times 0 and t. We call the steps between $I_{j-1}(t)$ and $I_j(t)-1$ the "jth period", and let $J_j(t) = I_j(t) -$

$I_{j-1}(t)$ be the length of the jth period. Note that $J_j(t)$ is geometrically distributed with parameter $(1/2)(1-e^{-t})$.

For each $j \geq 1$, define the event

$$A_j(t) = \{Z_i(0) > 0 \text{ and } Z_i(t) > 0 \ \ \forall i \in [I_{j-1}(t), I_j(t) - 1]\},$$

which says that our dynamical random walk is positive throughout the jth period at both time 0 and time t. The exception to this is for $A_1(t)$, where we shall only ask for it to be positive on $(I_0(t), I_1(t)-1] = [1, I_1(t)-1]$ as of course our random walk cannot be positive on the 0th step as it starts from zero. For each $i \geq 0$, let

$$W_i(t) = \frac{Z_i(0) + Z_i(t)}{2},$$

the average of the two walks $Z(0)$ and $Z(t)$. Note that, for each t, during odd periods the increments of $W_i(t)$ are equal to the increments of $Z_i(0)$; and during even periods, $W_i(t)$ is constant. (When we talk about increments we mean as i changes, keeping t fixed.)

When j is odd, define the event

$$A'_j(t) = \{W_i(t) > 0 \ \ \forall i \in [I_{j-1}(t), I_j(t) - 1]\}$$

that $W(t)$ is positive throughout the jth period. We make the same exception here for $j = 1$ as we did for $A_j(t)$. Note that, since $W_i(t)$ is the average of $Z_i(0)$ and $Z_i(t)$, if both of these are positive, then so is $W_i(t)$. That is, if j is odd, then $A_j(t) \subset A'_j(t)$.

Making the same comparison when j is even would not be useful since W is constant. Instead, when j is even, let $B_i^{(j)}(t)$, $i \geq 0$ be an independent simple random walk started from $W_{I_{j-1}(t)-1}(t)$ and define

$$A'_j(t) = \{B_i^{(j)}(t) \in (0, 2W_{I_{j-1}(t)-1}(t)) \ \ \forall i \in [1, J_j(t)]\}.$$

Figure 2-5 shows a realisation of $Z(0)$, $Z(t)$, $W(t)$, $B^{(2)}(t)$ and $B^{(4)}(t)$.

We need to rule out some unlikely events. Let

$$E_n^{\text{odd}}(t) = \{J_3(t) + J_5(t) + \ldots + J_{2\lfloor nt/8 \rfloor + 1}(t) \geq n/8\},$$

which we think of as the event that the odd periods (not including the first) are not

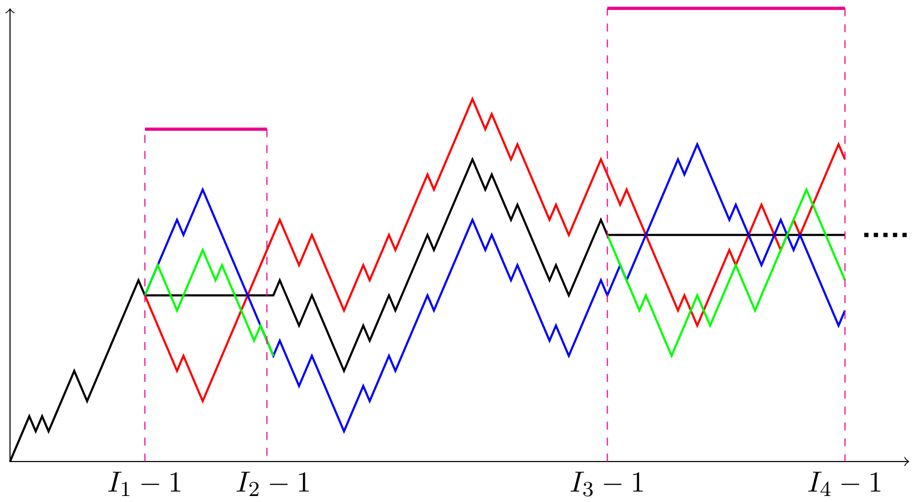

Figure 2-5: A realisation of $Z(0)$ and $Z(t)$ (blue/red), $W(t)$ (black), $B^{(2)}(t)$ and $B^{(4)}(t)$ (both green) for the first four periods.

too short,

$$E_n^{\text{even}}(t) = \{J_2(t) + J_4(t) + \ldots + J_{2\lfloor nt/8 \rfloor}(t) \geq n/8\},$$

which we think of as the event that the even periods are not too short,

$$E_n(t) = E_n^{\text{odd}}(t) \cap E_n^{\text{even}}(t)$$

the event that both the odd and even periods are not too short, and

$$E_n'(t) = \{I_{2\lfloor nt/8 \rfloor + 1}(t) \leq n\},$$

the event that we have at least $2\lfloor nt/8 \rfloor + 1$ periods before step n.

We note that for each j, when t is small $J_j(t)$ has expectation roughly $2/t$, so when n is large the above events should all occur with probability close to 1. The following lemma, which we prove later in the section, quantifies this more precisely.

Lemma 2.14. *There exists a constant $\delta > 0$ such that for any $t \in [0, 1]$ and $n \in \mathbb{N}$,*

$$\mathbb{P}(E_n(t)^c) + \mathbb{P}(E_n'(t)^c) \leq \exp(-\delta nt).$$

For now we will work on the event $E_n(t)$. Also define, for $k \in \mathbb{N}$,

$$V_k(t) = \bigcap_{j=1}^{k} A_j(t) \quad \text{and } V'_k(t) = \bigcap_{j=1}^{k} A'_j(t).$$

Our next result translates the probability that we want to bound, which is that of $V_k(t)$, into probabilities of events involving $W(t)$ and $B^{(j)}(t)$. The probabilities on the right are squared, reflecting the fact that we have two random walks (one at time 0 and another at time t) that must both stay positive. Apart from the first period, which is important to retain separately, only the even periods are included, since they are the ones on which the two random walks are mirrored.

Proposition 2.15. *For any $k, n \in \mathbb{N}$ with $n \geq 2k$ and any $t \in [0,1]$,*

$$\mathbb{P}\big(V_k(t) \cap E_n(t)\big) \leq \mathbb{P}\big(A'_1(t) \cap E_n(t)\big) \cdot \prod_{j=1}^{\lfloor k/2 \rfloor} \mathbb{P}\big(B_i^{(2j)}(t) > 0 \;\; \forall i \in [1, J_{2j}(t)] \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big)^2.$$

The proof of this result involves carefully separating out as much independence as possible between the different periods and applying the FKG inequality. Again we postpone the proof to later in the section in order to continue with our overarching proof of Proposition 2.8.

Next observe that since $B^{(j)}(t)$ is simply an independent random walk started from $W_{I_{j-1}(t)-1}(t)$, it has the same distribution as W itself over the $(j+1)$th period. This inspires our next proposition, which allows us to telescope the product from Proposition 2.15 back into a statement only about W.

Proposition 2.16. *For any $k, n \in \mathbb{N}$ with $n \geq 2k$ and any $t \in [0,1]$,*

$$\prod_{j=1}^{k} \mathbb{P}\big(B_i^{(2j)}(t) > 0 \;\; \forall i \in [1, J_{2j}(t)] \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big) = \frac{\mathbb{P}\big(\bigcap_{j=1}^{k+1} A'_{2j-1}(t) \cap E_n(t)\big)}{\mathbb{P}(A'_1(t) \cap E_n(t))}.$$

Combining Propositions 2.15 and 2.16, and then using elementary bounds, allows us to prove the following.

Proposition 2.17. *There exists a constant $C > 0$ such that for all (t, k, n) where*

$t \in (0,1]$, $n \in \mathbb{N}$ and $\lfloor k/2 \rfloor \geq \lfloor nt/8 \rfloor$, we have

$$\mathbb{P}\big(V_k(t) \cap E_n(t)\big) \leq \frac{C}{nt^{1/2}}.$$

Leaving the proof of Proposition 2.17 until later, we now observe that

$$\begin{aligned}
\mathbb{P}\big(P_n(0) \cap P_n(t)\big) &= \mathbb{P}\big(P_n(0) \cap P_n(t) \cap E_n(t) \cap E_n'(t)\big) \\
&\qquad + \mathbb{P}\big(P_n(0) \cap P_n(t) \cap (E_n(t)^c \cup E_n'(t)^c)\big) \\
&\leq \mathbb{P}\big(V_{2\lfloor nt/8 \rfloor+1}(t) \cap E_n(t)\big) + \mathbb{P}\big(P_n(0) \cap (E_n(t)^c \cup E_n'(t)^c)\big) \\
&= \mathbb{P}\big(V_{2\lfloor nt/8 \rfloor+1}(t) \cap E_n(t)\big) + \mathbb{P}\big(P_n(0)\big)\mathbb{P}\big(E_n(t)^c \cup E_n'(t)^c\big)
\end{aligned}$$

where the last equality used the independence of $Z(0)$ and the lengths of the periods at time t. By Proposition 2.17, the first term on the last line above is at most a constant times $1/(nt^{1/2})$, and by Corollary 2.6 and Lemma 2.14, the second term is at most a constant times $n^{-1/2}\exp(-\delta nt)$ for some constant $\delta > 0$. Thus

$$\mathbb{P}\big(P_n(0) \cap P_n(t)\big) \lesssim \frac{1}{nt^{1/2}} + \frac{1}{n^{1/2}}\exp(-\delta nt)$$

and so

$$\int_0^1 \frac{\mathbb{P}(P_n(0) \cap P_n(t))}{t^\gamma}\,\mathrm{d}t \lesssim \frac{1}{n}\int_0^1 t^{-1/2-\gamma}\,\mathrm{d}t + \frac{1}{n^{1/2}}\int_0^1 t^{-\gamma}e^{-\delta nt}\,\mathrm{d}t.$$

For $\gamma < 1/2$, the first integral on the right-hand side above is finite and the second integral (which can be approximated by integrating separately over $(0, 1/n]$ and $(1/n, 1)$) is of order $n^{\gamma-1}$. Therefore, for $\gamma < 1/2$,

$$\int_0^1 \frac{\mathbb{P}(P_n(0) \cap P_n(t))}{t^\gamma}\,\mathrm{d}t \lesssim n^{-1} + n^{\gamma-3/2} \asymp n^{-1}.$$

Recalling from the start of the section that

$$\mathbb{E}[\Phi_n^\alpha(\gamma)] \leq \frac{2}{\mathbb{P}(P_n^\alpha)^2}\int_0^1 \frac{\mathbb{P}(P_n(0) \cap P_n(t))}{t^\gamma}\,\mathrm{d}t,$$

and from Lemma 2.13 that for any $\alpha < 1/2$,

$$\mathbb{P}(P_n^\alpha) \asymp \frac{1}{\sqrt{n}},$$

we have for $\alpha, \gamma < 1/2$ that

$$\mathbb{E}[\Phi_n^\alpha(\gamma)] \lesssim 1.$$

This completes the proof of Proposition 2.8, subject to proving all of the intermediary results above.

Before we begin to prove these results, we will need another elementary lemma as an ingredient in the proof of Proposition 2.15.

Lemma 2.18. *If $(S_i,\ i \geq 0)$ is a simple symmetric random walk, then for any $x, y, k \in \mathbb{N}$,*

$$\mathbb{P}_x(S_i \in (0, 2y) \ \ \forall i \leq k) \leq \mathbb{P}_y(S_i \in (0, 2y) \ \ \forall i \leq k).$$

This is easily proved by induction. We include a proof later, but now proceed with the much more interesting proofs of Propositions 2.15 and 2.16. These proofs contain the main ideas of this chapter.

Proof of Proposition 2.15. Our first step is to move from $A_j(t)$ to $A'_j(t)$. To do so, we go via a third collection of events which we call $\tilde{A}_j(t)$. When j is odd, let $\tilde{A}_j(t) = A'_j(t)$. We have already mentioned that if j is odd, then

$$A_j(t) \subset A'_j(t) = \tilde{A}_j(t).$$

When j is even, define the event

$$\tilde{A}_j(t) = \{Z_i(0) \in (0, 2W_{I_{j-1}(t)-1}(t)) \ \ \forall i \in [I_{j-1}(t), I_j(t) - 1]\}.$$

We claim that when j is even, we also have $A_j(t) \subset \tilde{A}_j(t)$. Indeed, suppose that j is even. We show that if $\omega \notin \tilde{A}_j(t)$ then $\omega \notin A_j(t)$. If $\omega \notin \tilde{A}_j(t)$ then there exists $i \in [I_{j-1}(t), I_j(t) - 1]$ such that either $Z_i(0) \leq 0$, in which case clearly $\omega \notin A_j(t)$, or

$$Z_i(0) \geq 2W_{I_{j-1}(t)-1}(t) = Z_{I_{j-1}(t)-1}(0) + Z_{I_{j-1}(t)-1}(t).$$

Then

$$Z_i(0) - Z_{I_{j-1}(t)-1}(0) \geq Z_{I_{j-1}(t)-1}(t),$$

so since the increments of $Z_i(t)$ are the negative of the increments of $Z_i(0)$ during even periods,

$$Z_i(t) - Z_{I_{j-1}(t)-1}(t) \leq -Z_{I_{j-1}(t)-1}(t)$$

and therefore $Z_i(t) \leq 0$. Thus $\omega \notin A_j(t)$, establishing our claim. We deduce that, for any $k \in \mathbb{N}$,

$$A_1(t) \cap A_2(t) \cap \ldots \cap A_k(t) \subset \tilde{A}_1(t) \cap \tilde{A}_2(t) \cap \ldots \cap \tilde{A}_k(t). \tag{2.10}$$

Note that the increments of $Z_i(0)$ on even periods are independent of the whole process $W_i(t)$. Combining this fact with Lemma 2.18, we have

$$\mathbb{P}\big(\tilde{A}_1(t) \cap \tilde{A}_2(t) \cap \ldots \cap \tilde{A}_k(t) \big| \mathcal{F}_{I(t)}\big) \leq \mathbb{P}\big(A'_1(t) \cap A'_2(t) \cap \ldots \cap A'_k(t) \big| \mathcal{F}_{I(t)}\big) \tag{2.11}$$

for any $k \in \mathbb{N}$, where $\mathcal{F}_{I(t)} = \sigma(I_j(t), j \geq 0)$. Combining (2.10) and (2.11) and taking expectations to remove the conditioning, for any $k \in \mathbb{N}$ we have

$$\mathbb{P}(V_k(t) \cap E_n(t)) \leq \mathbb{P}(V'_k(t) \cap E_n(t)).$$

By repeatedly applying the definition of conditional probability, and then ignoring the odd terms for $j \geq 3$, we have

$$\begin{aligned} \mathbb{P}\big(V_k(t) \cap E_n(t)\big) &\leq \mathbb{P}\big(A'_1(t) \cap E_n(t)\big) \cdot \prod_{j=2}^{k} \mathbb{P}\big(A'_j(t) \,\big|\, V'_{j-1}(t) \cap E_n(t)\big) \\ &\leq \mathbb{P}\big(A'_1(t) \cap E_n(t)\big) \cdot \prod_{j=1}^{\lfloor k/2 \rfloor} \mathbb{P}\big(A'_{2j}(t) \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big). \end{aligned} \tag{2.12}$$

We now apply the FKG inequality (2.2). Recalling that $B^{(2j)}$ starts from $W_{I_{2j-1}(t)-1}$ and

$$\begin{aligned} A'_{2j}(t) &= \{B_i^{(2j)}(t) \in (0, 2W_{I_{2j-1}(t)-1}(t)) \ \ \forall i \in [1, J_{2j}(t)]\} \\ &= \{B_i^{(2j)}(t) > 0 \ \ \forall i \in [1, J_{2j}(t)]\} \\ &\qquad \cap \{B_i^{(2j)}(t) < 2W_{I_{2j-1}(t)-1}(t) \ \ \forall i \in [1, J_{2j}(t)]\}, \end{aligned}$$

and noting that the two events above are increasing and decreasing respectively, we get that

$$\begin{aligned} &\mathbb{P}\big(A'_{2j}(t) \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big) \\ &\qquad \leq \mathbb{P}\big(B_i^{(2j)}(t) > 0 \ \ \forall i \in [1, J_{2j}(t)] \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big) \\ &\qquad\quad \cdot \mathbb{P}\big(B_i^{(2j)}(t) < 2W_{I_{2j-1}(t)-1}(t) \ \ \forall i \in [1, J_{2j}(t)] \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big) \\ &\qquad = \mathbb{P}\big(B_i^{(2j)}(t) > 0 \ \ \forall i \in [1, J_{2j}(t)] \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big)^2, \end{aligned}$$

where the inequality comes from (2.2) and the equality follows from symmetry about $W_{I_{2j-1}(t)-1}(t)$ (recalling that $B_0^{(2j)}(t) = W_{I_{2j-1}(t)-1}(t)$). Substituting this into (2.12), we have shown that

$$\mathbb{P}\big(V_k(t) \cap E_n(t)\big) \leq \mathbb{P}\big(A_1'(t) \cap E_n(t)\big) \cdot \prod_{j=1}^{\lfloor k/2 \rfloor} \mathbb{P}\big(B_i^{(2j)}(t) > 0 \quad \forall i \in [1, J_{2j}(t)] \,\big|\, V_{2j-1}'(t) \cap E_n(t)\big)^2$$

as required. $\square$

The proof below of Proposition 2.16 is an induction argument that relies on the key idea that the walks $B_i^{(2j)}(t)$ are independent of the walk $W(t)$ but have the same step distribution. Thus we can just think about the next odd period of $W(t)$ rather than $B_i(t)$. This corresponds to contracting the walk W over the even periods where it is constant, and instead looking at the random walk generated by the odd periods. This can be seen in Figure 2-6.

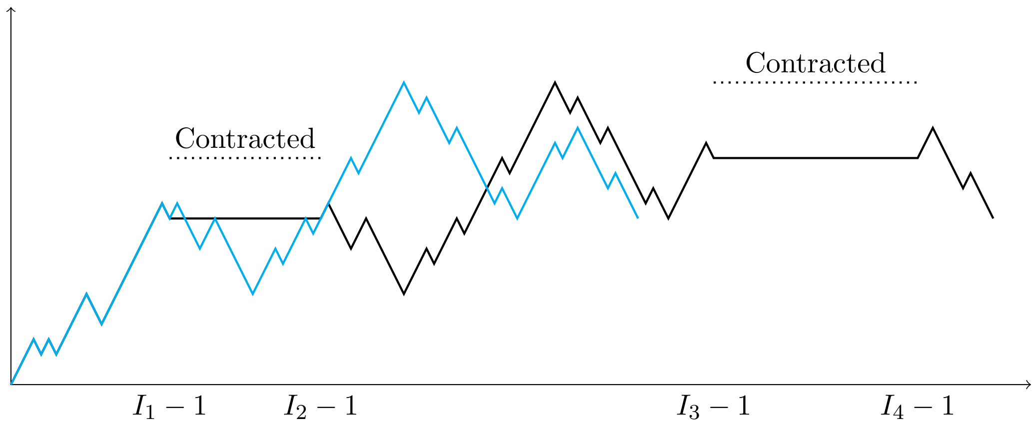

Figure 2-6: A realisation of $W(t)$ (black) and its contracted version (cyan) up until some step after the the fourth switch for $W(t)$.

Proof of Proposition 2.16. We work by induction on k. For $k = 1$, we have

$$\mathbb{P}\big(B_i^{(2)}(t) > 0 \quad \forall i \in [1, J_2(t)] \,\big|\, V_1'(t) \cap E_n(t)\big) = \frac{\mathbb{P}\big(\{B_i^{(2)}(t) > 0 \quad \forall i \in [1, J_2(t)]\} \cap A_1'(t) \cap E_n(t)\big)}{\mathbb{P}\big(A_1'(t) \cap E_n(t)\big)}.$$

On the event $A_1'(t) \cap E_n(t)$, the distribution of $(B_i^{(2)}(t))_{i \in [1, J_2(t)]}$ is identical to that of

$(W_{I_2(t)-1+i}(t))_{i\in[1,J_3(t)]}$, and therefore

$$\mathbb{P}\big(B_i^{(2)}(t)>0 \ \ \forall i\in[1,J_2(t)]\,\big|\,V_1'(t)\cap E_n(t)\big)=\frac{\mathbb{P}\big(A_3'(t)\cap A_1'(t)\cap E_n(t)\big)}{\mathbb{P}\big(A_1'(t)\cap E_n(t)\big)},$$

establishing the claim in the case $k=1$. The general case is very similar: assuming that the claim holds for $k-1$, we have

$$\begin{aligned}&\prod_{j=1}^{k}\mathbb{P}\big(B_i^{(2j)}(t)>0 \ \ \forall i\in[1,J_{2j}(t)]\,\big|\,V_{2j-1}'(t)\cap E_n(t)\big)\\&\quad=\frac{\mathbb{P}\big(\bigcap_{j=1}^{k}A_{2j-1}'(t)\cap E_n(t)\big)}{\mathbb{P}\big(A_1'(t)\cap E_n(t)\big)}\mathbb{P}\big(B_i^{(2k)}(t)>0 \ \ \forall i\in[1,J_{2k}(t)]\,\big|\,V_{2k-1}'(t)\cap E_n(t)\big).\end{aligned}$$

Considering the last term on the right-hand side above, we note that $B^{(2k)}(t)$ is independent of $A_{2j}'(t)$ given $A_{2j-1}'(t)$ for all $j<k$ and given $E_n(t)$, and therefore the above equals

$$\begin{aligned}&\frac{\mathbb{P}\big(\bigcap_{j=1}^{k}A_{2j-1}'(t)\cap E_n(t)\big)}{\mathbb{P}\big(A_1'(t)\cap E_n(t)\big)}\mathbb{P}\left(B_i^{(2k)}(t)>0 \ \ \forall i\in[1,J_{2k}(t)]\,\middle|\,\bigcap_{j=1}^{k}A_{2j-1}'(t)\cap E_n(t)\right)\\&\qquad\qquad=\frac{\mathbb{P}\big(\{B_i^{(2k)}(t)>0 \ \ \forall i\in[1,J_{2k}(t)]\}\cap\bigcap_{j=1}^{k}A_{2j-1}'(t)\cap E_n(t)\big)}{\mathbb{P}\big(A_1'(t)\cap E_n(t)\big)}.\end{aligned}$$

Provided that $2k\le n$, on the event $\bigcap_{j=1}^{k}A_{2j-1}'(t)\cap E_n(t)$, the law of $(B_i^{(2k)}(t))_{i\in[1,J_{2k}(t)]}$ is identical to that of $(W_{I_{2k}(t)-1+i}(t))_{i\in[1,J_{2k+1}(t)]}$, and therefore

$$\mathbb{P}\left(\left\{B_i^{(2k)}(t)>0 \ \ \forall i\in[1,J_{2k}(t)]\right\}\cap\bigcap_{j=1}^{k}A_{2j-1}'(t)\cap E_n(t)\right)=\mathbb{P}\left(\bigcap_{j=1}^{k+1}A_{2j-1}'(t)\cap E_n(t)\right)$$

which establishes the claim for k, completing the proof. $\square$

The proof of our third proposition in this section, Proposition 2.17, does not contain any major ideas; it simply combines the results above with some elementary approximations.

Proof of Proposition 2.17. Combining Propositions 2.15 and 2.16, we have

$$\mathbb{P}\big(V_k(t)\cap E_n(t)\big)\le\frac{\mathbb{P}\big(\bigcap_{j=1}^{\lfloor k/2\rfloor+1}A_{2j-1}'(t)\cap E_n(t)\big)^2}{\mathbb{P}(A_1'(t)\cap E_n(t))}.$$

Recalling that $A'_{2j-1}(t)$ requires that $W_i(t)$ is positive on the $(2j-1)$th period, whereas $W_i(t)$ is constant on even periods, we note that

$$\bigcap_{j=1}^{\lfloor k/2\rfloor+1} A'_{2j-1}(t) = \{W_i(t) > 0 \ \ \forall i \le I_{2\lfloor k/2\rfloor+1}(t) - 1\}$$

and therefore

$$\mathbb{P}\big(V_k(t) \cap E_n(t)\big) \le \frac{\mathbb{P}\big(\{W_i(t) > 0 \ \ \forall i \le I_{2\lfloor k/2\rfloor+1}(t) - 1\} \cap E_n(t)\big)^2}{\mathbb{P}(A'_1(t) \cap E_n(t))}.$$

Now, $W_i(t)$ is simply a simple symmetric random walk during odd periods, and constant on even periods. Thus the probability that it stays positive up to step $I_{2\lfloor k/2\rfloor+1}(t) - 1$ is exactly the probability that a simple symmetric random walk stays positive up to step $J_1(t) + J_3(t) + \ldots + J_{2\lfloor k/2\rfloor+1}(t) - 1$. We deduce that

$$\begin{aligned}
&\mathbb{P}\big(V_k(t) \cap E_n(t)\big) \\
&\qquad \le \frac{\mathbb{P}\big(\{Z_i(t) > 0 \ \ \forall i \le J_1(t) + J_3(t) + \ldots + J_{2\lfloor k/2\rfloor+1}(t) - 1\} \cap E_n(t)\big)^2}{\mathbb{P}(A'_1(t) \cap E_n(t))} \\
&\qquad \le \frac{\mathbb{P}\big(Z_i(t) > 0 \ \ \forall i \le J_1(t) + J_3(t) + \ldots + J_{2\lfloor k/2\rfloor+1}(t) - 1 \,\big|\, E_n(t)\big)^2}{\mathbb{P}\big(A'_1(t) \,\big|\, E_n(t)\big)}.
\end{aligned}$$

On the event $E_n(t) \subset E_n^{\mathrm{odd}}(t)$, we have

$$J_1(t) + J_3(t) + \ldots + J_{2\lfloor nt/8\rfloor+1}(t) - 1 \ge J_3(t) + J_5(t) + \ldots + J_{2\lfloor nt/8\rfloor+1}(t) \ge n/8,$$

and therefore for any k such that $\lfloor k/2\rfloor \ge \lfloor nt/8\rfloor$,

$$\mathbb{P}\big(V_k(t) \cap E_n(t)\big) \le \frac{\mathbb{P}\big(Z_i(t) > 0 \ \ \forall i \le n/8\big)^2}{\mathbb{P}\big(A'_1(t) \,\big|\, E_n(t)\big)} = \frac{\mathbb{P}\big(Z_i(0) > 0 \ \ \forall i \le n/8\big)^2}{\mathbb{P}(A'_1(t))}, \tag{2.13}$$

where the equality holds by stationarity of $Z(t)$ and the independence of $A'_1(t)$ and $E_n(t)$ (since $E_n(t)$ only involves periods 2 and later). We know from Corollary 2.6 that

$$\mathbb{P}\big(Z_i(0) > 0 \ \ \forall i \le n/8\big) \asymp n^{-1/2},$$

and we claim that

$$\mathbb{P}(A'_1(t)) \gtrsim t^{1/2}.$$

To see this, note that $I_1(t)$ is independent of $Z(0)$, so

$$\begin{aligned}\mathbb{P}(A_1'(t)) &= \mathbb{P}(Z_i(0) > 0 \ \ \forall i = 1, \dots, I_1(t)) \\ &\geq \mathbb{P}\Big(I_1(t) \leq \Big\lceil \frac{4}{1-e^{-t}} \Big\rceil\Big)\mathbb{P}\Big(Z_i(0) > 0 \ \ \forall i = 1, \dots, \Big\lceil \frac{4}{1-e^{-t}} \Big\rceil\Big).\end{aligned}$$

But by Markov's inequality

$$\mathbb{P}\Big(I_1(t) \leq \Big\lceil \frac{4}{1-e^{-t}} \Big\rceil\Big) = 1 - \mathbb{P}\Big(I_1(t) > \Big\lceil \frac{4}{1-e^{-t}} \Big\rceil\Big) \geq 1 - \frac{1-e^{-t}}{4}\mathbb{E}[I_1(t)] = 1 - \frac{1}{2} = \frac{1}{2};$$

and by Corollary 2.6,

$$\mathbb{P}\Big(Z_i(0) > 0 \ \ \forall i = 1, \dots, \Big\lceil \frac{4}{1-e^{-t}} \Big\rceil\Big) \asymp (1-e^{-t})^{1/2} \asymp t^{1/2},$$

which establishes the claim. Substituting our approximations into (2.13), we have shown that for any k such that $\lfloor k/2 \rfloor \geq \lfloor nt/8 \rfloor$,

$$\mathbb{P}\big(V_k(t) \cap E_n(t)\big) \leq \frac{C}{nt^{1/2}}$$

as required. □

We now proceed with the proofs of our minor lemmas.

Proof of Lemma 2.13. Recalling that

$$P_n = \{Z_i > 0 \ \ \forall i = 1, \dots, n\} \quad \text{and} \quad P_n^\alpha = \big\{Z_i \geq i^\alpha \ \ \forall i = 1, \dots, n\big\},$$

we use the fact that $\mathbb{P}(P_n^\alpha) = \mathbb{P}(P_n^\alpha | P_n)\mathbb{P}(P_n)$. From Corollary 2.6 we know that $\mathbb{P}(P_n) \asymp n^{-1/2}$. It therefore suffices to show that $\mathbb{P}(P_n^\alpha) \asymp \mathbb{P}(P_n)$ for any $\alpha < 1/2$. Fix $\alpha' \in (\alpha, 1/2)$. We apply [42, Theorem 2], which says that we may choose $\delta > 0$ such that

$$\mathbb{P}(Z_i \geq \delta i^{\alpha'} \ \ \forall i = 1, \dots, n) \geq \mathbb{P}(P_n)/2.$$

Choose k such that $\delta i^{\alpha'} \geq i^\alpha$ for all $i \geq k$. Then

$$\begin{aligned}\mathbb{P}(Z_i \geq i^\alpha \ \ \forall i = 1, \dots, n) &\geq \mathbb{P}(Z_i = i \ \ \forall i = 1, \dots, k;\ Z_i \geq i^\alpha \ \ \forall i = k+1, \dots, n) \\ &\geq \mathbb{P}(Z_i = i \ \ \forall i = 1, \dots, k;\ Z_i \geq \delta i^{\alpha'} \ \ \forall i = k+1, \dots, n) \\ &= 2^{-k}\mathbb{P}(Z_i \geq \delta(i+k)^{\alpha'} - k \ \ \forall i = 1, \dots, n-k) \\ &\geq 2^{-k}\mathbb{P}(Z_i \geq \delta i^{\alpha'} \ \ \forall i = 1, \dots, n) \geq 2^{-(k+1)}\mathbb{P}(P_n),\end{aligned}$$

which completes the proof. $\square$

Proof of Lemma 2.14. We begin by considering $E_n^{\text{odd}}(t)$. In order for $E_n^{\text{odd}}(t)^c$ to occur, the sum of $\lfloor nt/8 \rfloor$ independent geometric random variables of parameter $(1-e^{-t})/2$ must be smaller than $n/8$; which is equivalent to a Binomial random variable of parameters $(\lceil n/8 \rceil, (1-e^{-t})/2)$ being larger than $\lfloor nt/8 \rfloor$. Letting X be such a random variable, we have

$$\begin{aligned}\mathbb{E}[e^{(\log 2)X}] = \Big((1+e^{-t})/2 + (1-e^{-t})\Big)^{\lceil n/8 \rceil} &= \Big(1 + (1-e^{-t})/2\Big)^{\lceil n/8 \rceil} \\ &\le (1+t/2)^{\lceil n/8 \rceil} \le e^{(n/8+1)t/2},\end{aligned}$$

so

$$\begin{aligned}\mathbb{P}(Y \ge \lfloor nt/8 \rfloor) \le \mathbb{E}[e^{(\log 2)Y}]e^{-(\log 2)\lfloor nt/8 \rfloor} &\le e^{(n/8+1)t/2-(\log 2)(nt/8-1)} \\ &\le 2e^{1/2}e^{-(2\log 2-1)nt/16}.\end{aligned}$$

This proves the required decay for $\mathbb{P}(E_n^{\text{odd}}(t)^c)$, and $\mathbb{P}(E_n^{\text{even}}(t)) = \mathbb{P}(E_n^{\text{odd}}(t))$. The proof for $\mathbb{P}(E_n'(t)^c)$ is very similar. Noting that $I_j(t)$ is a sum of j independent Geometric random variables of parameter $(1-e^{-t})/2$, we have that $\mathbb{P}(E_n'(t)^c) \le \mathbb{P}(Y \le 2\lfloor nt/8 \rfloor + 1)$ where $Y \sim Bin(n, (1-e^{-t}/2))$. Now

$$\mathbb{E}[e^{-(\log 2)Y}] = \Big((1+e^{-t})/2 + (1-e^{-t})/4\Big)^n = \Big((1/4)(3+e^{-t})\Big)^n \le (1-t/8)^n \le e^{-nt/8}$$

where we've used that $1-t \le e^{-t} \le 1-t/2$ for $t \in [0,1]$. Using the Chernoff bound again we get

$$\begin{aligned}\mathbb{P}(Y \le 2\lfloor nt/8 \rfloor + 1) \le \mathbb{E}[e^{-(\log 2)Y}]e^{(\log 2)(2\lfloor nt/8 \rfloor+1)} &\le 2e^{-(nt/8)+(\log 2)(nt/4)} \\ &= 2e^{-(2\log 2-1)nt/8}.\end{aligned}$$ $\square$

Proof of Lemma 2.18. Fix $y \in \mathbb{N}$ and let

$$p_{x,k} = \mathbb{P}_x(S_i \in (0,2y) \quad \forall i \le k).$$

We claim, by induction on k, that $p_{x,k}$ is non-decreasing in x for $x \le y$. By symmetry this is enough to prove the lemma. Clearly the claim holds for $k = 0$. For general k, if $x = y$ then by symmetry

$$p_{y,k+1} = \frac{1}{2}p_{y-1,k} + \frac{1}{2}p_{y+1,k} = p_{y-1,k}$$

which is larger than $p_{y-1,k+1}$ by definition. On the other hand if $x < y$, then by the induction hypothesis,

$$p_{x,k+1} = \frac{1}{2}p_{x-1,k} + \frac{1}{2}p_{x+1,k} \geq \frac{1}{2}p_{x-2,k} + \frac{1}{2}p_{x,k} = p_{x-1,k+1}.$$

This completes the proof of our final lemma in this section, and therefore the proof of Proposition 2.8. □

2.6 Proof of Proposition 2.12: influences of P_n

In this section we give estimates on the influence of each bit $m = 1, 2, \ldots, n$ on the event P_n. Proposition 2.12 stated that for $m = 1, \ldots, n$,

$$\mathcal{I}_m(P_n) \asymp \frac{n-m+1}{n^{3/2}},$$

where $\mathcal{I}_m(P_n)$ is the probability that the mth bit is pivotal for P_n, and it will be our aim to prove this. We will keep n fixed and say "m is pivotal" as shorthand for "m is pivotal for P_n".

2.6.1 Translating $\mathcal{I}_m(P_n)$ into elementary properties of the random walk

To reduce the amount of work we will take advantage of the fact that

$$\mathcal{I}_m(P_n) = \mathbb{P}(m \text{ is pivotal}) = 2\mathbb{P}(\{m \text{ is pivotal}\} \cap P_n), \tag{2.14}$$

which holds since the event that m is pivotal is independent of the value of X_m:

$$\begin{aligned}
&\mathbb{P}(\{m \text{ is pivotal}\} \cap P_n) \\
&= \mathbb{P}(\{m \text{ is pivotal}\} \cap \{X_m = 1\} \cap P_n) + \mathbb{P}(\{m \text{ is pivotal}\} \cap \{X_m = -1\} \cap P_n) \\
&= \mathbb{P}(\{m \text{ is pivotal}\} \cap \{X_m = -1\} \cap P_n^c) + \mathbb{P}(\{m \text{ is pivotal}\} \cap \{X_m = 1\} \cap P_n^c) \\
&= \mathbb{P}(\{m \text{ is pivotal}\} \cap P_n^c).
\end{aligned}$$

We now write down an explicit condition for the event $\{m \text{ is pivotal}\} \cap P_n$ to occur. We claim that for $m = 1, 2, \ldots, n$,

$$\{m \text{ is pivotal}\} \cap P_n = \{Z_i > 0 \;\; \forall i = 1, \ldots, n\} \cap \Big\{ \max_{m \leq i \leq n} Z_i \geq 2Z_{m-1} \Big\}. \tag{2.15}$$

In words, m is pivotal and P_n holds if and only if Z stays positive for the first n steps, and hits $2Z_{m-1}$ between steps m and n.

To see why this is true, call the path of Z up to step $m-1$ the *first portion* of the walk, and the path from step m to step n the *second portion*. Of course P_n entails that both portions remain positive. In order for m to be pivotal, we also need that when we change the sign of the mth bit, and therefore reflect the second portion of the path about Z_{m-1}, the second portion no longer remains positive. This holds if and only if the second portion (before reflection) hits $2Z_{m-1}$. See Figure 2-7.

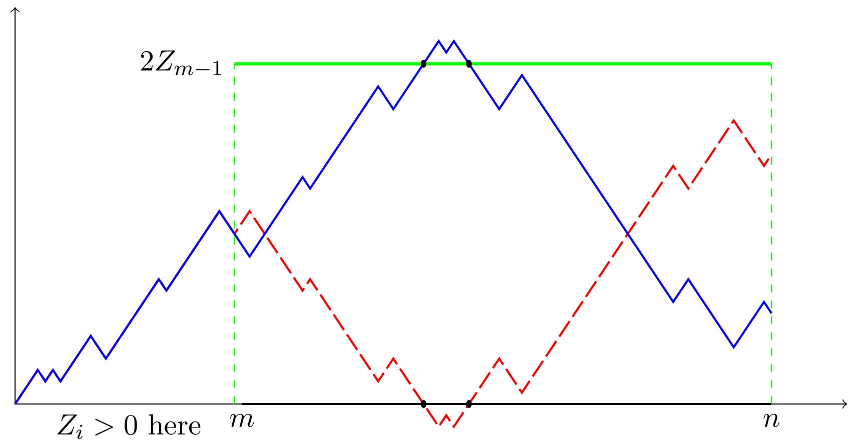

Figure 2-7: A realisation of Z with and without the mth bit flipped (dashed red / solid blue). The black dots show the points at which the walks hits one of the two barriers at 0 or $2Z_{m-1}$, which is the key to pivotality.

If $m = 1$ then trivially $Z_{m-1} = 0$, so (2.15) reduces to

$$\{1 \text{ is pivotal}\} \cap P_n = \{Z_i > 0 \ \ \forall i = 1, \ldots, n\}.$$

Thus, by Corollary 2.6, $\mathbb{P}(\{1 \text{ is pivotal}\} \cap P_n)$ is of order $n^{-1/2}$. Proposition 2.12 therefore holds for $m = 1$ and we may assume that from now on $m \geq 2$.

Returning to (2.15) in the case $m \geq 2$, the next step is to split the event that m is pivotal over the possible values of Z_{m-1}. Writing $\mathbb{P}_z$ for the probability measure under which our walk starts from z instead of 0, by (2.14) and (2.15)

$$\mathcal{I}_m(P_n) = 2 \sum_{z=1}^{m-1} \mathbb{P}_0\Big(\min_{1 \leq i \leq m-1} Z_i > 0,\, Z_{m-1} = z \Big) \\ \cdot \mathbb{P}_z\Big(\big\{ \min_{i \leq n-m+1} Z_i > 0 \big\} \cap \big\{ \max_{i \leq m-n+1} Z_i \geq 2z \big\} \Big).$$

By the ballot theorem [4] (or see [1] for a thorough introduction), the probability that a simple symmetric random walk starting from 0 stays positive up to step $m-1$ and finishes at z at the $(m-1)$-st step is $z/(m-1)$ times the probability that the random walk finishes at z at the $(m-1)$-st step; thus

$$\mathcal{I}_m(P_n) = 2\sum_{z=1}^{m-1} \frac{z}{m-1}\mathbb{P}_0(Z_{m-1} = z) \cdot \mathbb{P}_z\Big(\big\{\min_{i \le n-m+1} Z_i > 0\big\} \cap \big\{\max_{i \le m-n+1} Z_i \ge 2z\big\}\Big). \tag{2.16}$$

2.6.2 A lower bound on the influences of P_n

Define the events

$$L = L(m,n) = \big\{\min_{1 \le i \le n-m+1} Z_i > 0\big\} \quad \text{and} \quad U = U(m,n,z) = \big\{\max_{1 \le i \le n-m+1} Z_i \ge 2z\big\}. \tag{2.17}$$

Let

$$l(m,n) = \Big\lfloor \frac{\sqrt{n-m+1}}{2} \Big\rfloor \wedge \Big\lfloor \frac{\sqrt{m-1}}{2} \Big\rfloor.$$

We want to bound $\mathbb{P}_z(L \cap U)$ from below when $z \le l(m,n)$. The following corollary of Lemmas 2.3 and 2.5 will be useful.

Corollary 2.19. *If $0 \le z \le \sqrt{n-m+1}$ with $0 \le m \le n$ then as n grows large:*

$$\mathbb{P}_z(L(m,n)) \asymp \frac{z+1}{\sqrt{n-m+1}}$$

and if $0 \le z \le l(m,n)$ with $0 \le m \le n$ then for large n

$$\mathbb{P}_z(U(m,n,z)) \asymp 1.$$

Proof. From Lemma 2.5,

$$\mathbb{P}_z(L) = \mathbb{P}_z(Z_i > 0 \ \ \forall i \le n-m+1) = \mathbb{P}_0(Z_{n-m+1} \in [-z+1, z]),$$

and by Lemma 2.3, this is of order

$$\sum_{i=-z+1}^{z} \frac{1}{\sqrt{n-m+1}} \exp\Big(-\frac{i^2}{2(n-m+1)}\Big).$$

The first part of the result now follows from the fact that $z \le \sqrt{n-m+1}$. The second part is very similar: using Lemmas 2.5 and 2.3,

$$\begin{aligned}\mathbb{P}_z(U) = 1 - \mathbb{P}_z(L) = 1 - \mathbb{P}_0(Z_{n-m+1} \in [-z+1, z]) \geq \mathbb{P}_0(Z_{n-m+1} \geq z+1) \\ \geq \sum_{y=z+1}^{\lfloor\sqrt{n-m+1}\rfloor} \mathbb{P}_0(Z_{n-m+1} = y) \gtrsim \sum_{y=z+1}^{\lfloor\sqrt{n-m+1}\rfloor} \frac{1}{\sqrt{n-m+1}} \asymp 1\end{aligned}$$

and clearly $\mathbb{P}_z(U) \leq 1$ so the proof is complete. □

Lemma 2.20. *For $z \in [0, l(m,n)]$, we have*

$$\mathbb{P}_z\Big(L(m,n) \cap U(m,n,z)\Big) \gtrsim \frac{z}{\sqrt{n-m+1}}.$$

Proof. We would like to use the FKG inequality. Unfortunately, neither L nor U is either increasing or decreasing as a function of X. However, if we replace the switch random walk Z with the compass random walk Y, setting

$$L' = \big\{\min_{i \leq n-m+1} Y_i > 0\big\} \quad \text{and} \quad U' = \big\{\max_{i \leq n-m+1} Y_i \geq 2z\big\},$$

then L' and U' are both increasing. Thus the FKG inequality (2.1) tells us that

$$\mathbb{P}_z(L' \cap U') \geq \mathbb{P}_z(L')\mathbb{P}_z(U')$$

and since Y and Z have the same distribution,

$$\mathbb{P}_z(L \cap U) = \mathbb{P}_z(L' \cap U') \geq \mathbb{P}_z(L')\mathbb{P}_z(U') = \mathbb{P}_z(L)\mathbb{P}_z(U).$$

The result now follows from Corollary 2.19. □

Substituting the result of Lemma 2.20 into (2.16) gives that

$$\begin{aligned}\mathcal{I}_m(P_n) &\geq 2\sum_{z=1}^{l(m,n)} \frac{z}{m-1}\mathbb{P}_0(Z_{m-1} = z) \cdot \mathbb{P}_z\Big(\big\{\min_{i \leq n-m+1} Z_i > 0\big\} \cap \big\{\max_{i \leq m-n+1} Z_i \geq 2z\big\}\Big) \\ &\gtrsim \sum_{z=1}^{l(m,n)} \frac{z}{m-1}\mathbb{P}_0(Z_{m-1} = z) \cdot \frac{z}{\sqrt{n-m+1}}.\end{aligned}$$

Applying Lemma 2.3 again tells us that for $z \in [1, l(m,n)]$, we have $\mathbb{P}_0(Z_{m-1} = z) \asymp$

$(m-1)^{-1/2}$; so

$$\mathcal{I}_m(P_n) \gtrsim \sum_{z=1}^{l(m,n)} \frac{z}{m-1} \cdot \frac{1}{\sqrt{m-1}} \cdot \frac{z}{\sqrt{n-m+1}} \asymp \frac{l(m,n)^3}{(m-1)^{3/2}(n-m+1)^{1/2}}.$$

If $m \leq n/2$, then the right-hand side above is of order $n^{-1/2}$, and if $m > n/2$, it is of order $(n-m+1)/n^{3/2}$. In either case this completes the proof of the lower bound in Proposition 2.12.

2.6.3 An upper bound on the influences of P_n

We will now bound (2.16) from above. This direction is more challenging as we need to consider the entire sum; for the lower bound we could restrict to just the values of z that gave the biggest contribution. We recall the definitions of L and U from (2.17).

Some of the following arguments have been streamlined since their publication in [40] as we only need to focus on the case that $m \geq n/2$. This is due to the observation that if $m < n/2$ then by Corollary 2.6 and (2.14)

$$\mathcal{I}_m(P_n) = 2\mathbb{P}(\{m \text{ is pivotal}\} \cap P_n) \leq 2\mathbb{P}(P_n) \asymp n^{-1/2} \asymp (n-m+1)n^{-3/2}$$

so we obtain the required upper bound. Now let $m \geq n/2$. As part of our proof we will have to bound several sums of the following form.

Lemma 2.21. *There exists $K > 0$ such that for all $c \in \mathbb{N}$ and $r \geq 0$*

$$\sum_{z=0}^{\infty} (z+1)^r \exp\Big(-\frac{z^2}{c}\Big) \leq Kc^{(r+1)/2}.$$

Proof. Letting $C = \lceil \sqrt{c} \rceil$, we have

$$\begin{aligned}
\sum_{z=0}^{\infty} (z+1)^r \exp\Big(-\frac{z^2}{c}\Big) &= \sum_{k=0}^{\infty} \sum_{z=kC}^{(k+1)C-1} (z+1)^r \exp\Big(-\frac{z^2}{c}\Big) \\
&\leq \sum_{k=0}^{\infty} C((k+1)C)^r \exp\Big(-\frac{k^2C^2}{c}\Big) \\
&\leq C^{r+1} \sum_{k=0}^{\infty} (k+1)^r \exp(-k^2) \asymp C^{r+1}. \qquad \square
\end{aligned}$$

Let $M = \lfloor (m-1)^{3/4} \rfloor$. We begin our upper bound on (2.16) by splitting the sum depending on whether z is larger or smaller than M: from (2.16),

$$\begin{aligned}\mathcal{I}_m(P_n) &= 2\sum_{z=1}^{M}\frac{z}{m-1}\mathbb{P}_0(Z_{m-1}=z)\mathbb{P}_z(L\cap U)\\ &\qquad + 2\sum_{z=M+1}^{m-1}\frac{z}{m-1}\mathbb{P}_0(Z_{m-1}=z)\mathbb{P}_z(L\cap U)\\ &\leq 2\sum_{z=1}^{M}\frac{z}{m-1}\mathbb{P}_0(Z_{m-1}=z)\mathbb{P}_z(L\cap U) + 2\sum_{z=M+1}^{m-1}\mathbb{P}_0(Z_{m-1}=z)\mathbb{P}_z(L). \quad (2.18)\end{aligned}$$

We label the two sums in (2.18) by (2.18 i) and (2.18 ii).

Addressing the second sum first, we note that $\mathbb{P}_z(L)$ is increasing in z, so

$$(2.18\,\text{ii}) \leq 2\mathbb{P}_{m-1}(L)\sum_{z=M+1}^{m-1}\mathbb{P}_0(Z_{m-1}=z) = 2\mathbb{P}_{m-1}(L)\mathbb{P}_0(Z_{m-1}>M).$$

By Lemma 2.4 with $x=M$, we have

$$\mathbb{P}_0(Z_{m-1}>M) \leq \exp(-(m-1)^{1/2}/2).$$

Using the trivial bound $\mathbb{P}_{m-1}(L)\leq 1$, we have that

$$(2.18\,\text{ii}) \lesssim \exp(-(m-1)^{1/2}/2).$$

As $m \geq n/2$, one can check that the above is at most a constant times $(n-m+1)n^{-3/2}$, as required. It thus remains to bound (2.18 i).

To do this we split it again depending on whether z exceeds $\lfloor (n-m+1)^{1/2}\rfloor$. If it does not, we bound $\mathbb{P}_z(L\cap U)$ above by $\mathbb{P}_z(L)$ and apply Lemma 2.3 and Corollary 2.19. Nothing that $M' = \lfloor (n-m+1)^{1/2}\rfloor \leq M$, we obtain

$$\begin{aligned}&\sum_{z=1}^{M'}\frac{z}{m-1}\mathbb{P}_0(Z_{m-1}=z)\mathbb{P}_z(L\cap U)\\ &\qquad\qquad \leq \sum_{z=1}^{M'}\frac{z}{m-1}\mathbb{P}_0(Z_{m-1}=z)\mathbb{P}_z(L)\\ &\qquad\qquad \asymp \sum_{z=1}^{M'}\frac{z}{m-1}\frac{1}{(m-1)^{1/2}}e^{-z^2/(2(m-1))}\frac{z+1}{(n-m+1)^{1/2}}. \quad (2.19)\end{aligned}$$

As $m \geq n/2$,

$$\sum_{z=1}^{M'} z(z+1)e^{-\frac{z^2}{2(m-1)}} \leq \sum_{z=1}^{\lfloor (n-m+1)^{1/2} \rfloor} z(z+1) \asymp (n-m+1)^{3/2}.$$

Applying this bound to (2.19) gives that

$$\sum_{z=1}^{M'} \frac{z}{m-1}\mathbb{P}_0(Z_{m-1}=z)\mathbb{P}_z(L\cap U) \lesssim \frac{(n-m+1)^{3/2}}{(m-1)^{3/2}(n-m+1)^{1/2}} \lesssim \frac{n-m+1}{n^{3/2}}, \quad (2.20)$$

as required.

When $z > (n-m+1)^{1/2}$ then we bound $\mathbb{P}_z(L \cap U)$ above by $\mathbb{P}_z(U)$ instead of $\mathbb{P}_z(L)$. Applying Lemma 2.3, we have

$$\begin{aligned}\sum_{z=M'+1}^{M} \frac{z}{m-1}\mathbb{P}_0(Z_{m-1}=z)\mathbb{P}_z(L\cap U) &\lesssim \sum_{z=M'+1}^{M} \frac{z}{m-1}\frac{1}{(m-1)^{1/2}}e^{-z^2/(2(m-1))}\mathbb{P}_z(U) \\ &\leq \sum_{z=M'+1}^{M} \frac{z}{m-1}\frac{1}{(m-1)^{1/2}}\mathbb{P}_z(U),\end{aligned}$$

and by Lemmas 2.5 and 2.4,

$$\begin{aligned}\mathbb{P}_z(U) &= 1-\mathbb{P}_z(Z_i < 2z \quad \forall i \leq n-m+1) \\ &= 1-\mathbb{P}(Z_{n-m+1} \in [-z+1, z]) \leq 2\mathbb{P}(Z_{n-m+1} \geq z) \leq 2\exp\Big(-\frac{z^2}{2(n-m+1)}\Big).\end{aligned}$$

Thus

$$\sum_{z=M'+1}^{M} \frac{z}{m-1}\mathbb{P}_0(Z_{m-1}=z)\mathbb{P}_z(L\cap U) \leq \sum_{z=0}^{\infty} \frac{2z}{(m-1)^{3/2}}e^{-z^2/(2(n-m+1))}. \qquad (2.21)$$

By Lemma 2.21 (noting $m \geq n/2$), this is of order at most $(n-m+1)/n^{3/2}$. Combining this with (2.20), we have shown that

$$(2.18\,\text{i}) \lesssim \frac{n-m+1}{n^{3/2}},$$

which completes the proof of Proposition 2.12.

2.7 Proofs of Lemmas 2.7, 2.9 and 2.10

To complete our proof of the lower bound on the Hausdorff dimension of $\mathcal{E}$ outlined in Section 2.4, we need several technical lemmas. In this section we prove those results, beginning with Lemma 2.7, which is based on [44, Lemma 6.2].

Proof of Lemma 2.7. If we let μ_n be the measure on $[0,1]$ given by

$$\mu_n(A) = \frac{1}{\mathbb{P}(A_n)} \int_A \mathbb{1}_{A_n(t)} \,\mathrm{d}t,$$

then noting that μ_n is supported on $\bar{T}_n$, [44, Lemma 6.2] gives a sufficient condition for the Hausdorff dimension of $\bigcap_n \bar{T}_n$ to be at least γ. This condition is that there exists a finite positive constant c such that for infinitely many n,

$$\mu_n([0,1]) \geq 1/c \quad \text{and} \quad \int_0^1 \int_0^1 |t-s|^{-\gamma} \,\mathrm{d}\mu_n(s) \,\mathrm{d}\mu_n(t) \leq c.$$

We start by bounding $\mu_n([0,1])$ from below. By the Paley-Zygmund inequality,

$$\mathbb{P}\Big(\mu_n([0,1]) \geq \frac{1}{2}\mathbb{E}[\mu_n([0,1])]\Big) \geq \frac{\mathbb{E}[\mu_n([0,1])]^2}{4\mathbb{E}[\mu_n([0,1])^2]}. \tag{2.22}$$

By Fubini's theorem and stationarity,

$$\mathbb{E}[\mu_n([0,1])] = \frac{1}{\mathbb{P}(A_n)} \int_0^1 \mathbb{P}(A_n(t)) \,\mathrm{d}t = \frac{1}{\mathbb{P}(A_n)} \int_0^1 \mathbb{P}(A_n) \,\mathrm{d}t = 1.$$

Also, for any $\gamma \in [0,1)$,

$$\mathbb{E}[\mu_n([0,1])^2] = \mathbb{E}\Big[\frac{1}{\mathbb{P}(A_n)^2} \int_0^1 \int_0^1 \mathbb{1}_{A_n(s)} \mathbb{1}_{A_n(t)} \,\mathrm{d}s \,\mathrm{d}t\Big] = \mathbb{E}[\Phi_n(0)] \leq \mathbb{E}[\Phi_n(\gamma)].$$

Substituting these estimates into (2.22), we have

$$\mathbb{P}(\mu_n([0,1]) \geq 1/2) \geq \frac{1}{4\mathbb{E}[\Phi_n(\gamma)]}$$

so fixing γ to be the value in the statement of the lemma and letting $S = \sup_n \mathbb{E}[\Phi_n(\gamma)]$, we have

$$\inf_n \mathbb{P}(\mu_n([0,1]) \geq 1/2) \geq \frac{1}{4S}.$$

Now note that

$$\Phi_n(\gamma) = \int_0^1 \int_0^1 |t-s|^{-\gamma} \, \mathrm{d}\mu_n(s) \, \mathrm{d}\mu_n(t),$$

so the second part of our desired condition requires us to show that $\Phi_n(\gamma) \le c$ for some constant c and infinitely many n. By Markov's inequality,

$$\sup_n \mathbb{P}(\Phi_n(\gamma) > 8S^2) \le \sup_n \frac{\mathbb{E}[\Phi_n(\gamma)]}{8S^2} = \frac{1}{8S},$$

and therefore

$$\begin{aligned} &\inf_n \mathbb{P}(\mu_n([0,1]) \ge 1/2 \text{ and } \Phi_n(\gamma) \le 8S^2) \\ &\qquad\qquad \ge \inf_n \mathbb{P}(\mu_n([0,1]) \ge 1/2) - \sup_n \mathbb{P}(\Phi_n(\gamma) > 8S^2) \ge \frac{1}{8S}. \end{aligned}$$

By Fatou's lemma we deduce that

$$\begin{aligned} &\mathbb{P}(\mu_n([0,1]) \ge 1/2 \text{ and } \Phi_n(\gamma) \le 8S^2 \text{ for infinitely many } n) \\ &\qquad = \mathbb{E}[\mathbb{1}_{\limsup_{n\to\infty}\{\mu_n([0,1])\ge 1/2 \text{ and } \Phi_n(\gamma)\}}] \\ &\qquad = \mathbb{E}[\limsup_{n\to\infty} \mathbb{1}_{\{\mu_n([0,1])\ge 1/2 \text{ and } \Phi_n(\gamma)\}}] \\ &\qquad \ge \limsup_{n\to\infty} \mathbb{E}[\mathbb{1}_{\{\mu_n([0,1])\ge 1/2 \text{ and } \Phi_n(\gamma)\}}] \\ &\qquad \ge \inf_n \mathbb{P}(\mu_n([0,1]) \ge 1/2 \text{ and } \Phi_n(\gamma) \le 8S^2) \ge \frac{1}{8S}, \end{aligned}$$

and the proof is complete. □

Our proof of Lemma 2.9 is based on the equivalent result for percolation by Häggström, Peres and Steif [36, Lemma 3.2].

Proof of Lemma 2.9. Recall that for each j, $(N_j(t), t \ge 0)$ is a Poisson process of rate 1 that decides when X_j rerandomises. For $i \ge 0$, let $\tau_j^{(i)} = \inf\{t \ge 0 : N_j(t) = i\}$, the time of the ith rerandomisation of X_j.

Fix i and j. Since each step of the random walk evolves (in time) independently, almost surely at time $\tau_j^{(i)}$ the random walk hits both 0 and $2Z_{j-1}(\tau_j^{(i)})$ after step j; thus for large enough n, the random walk hits 0 before step n regardless of the state of step j. The random walk therefore also falls below the line $i \mapsto i^\alpha$ before step n (for large enough n), regardless of the state of step j. That is, almost surely, $\tau_j^{(i)} \notin \bar{T}_n^\alpha \setminus T_n^\alpha$ for all large n.

However, since the system only changes when one of the X_j rerandomises, for each $\alpha \geq 0$ and $n \in \mathbb{N}$ we have

$$\bar{T}_n^\alpha \setminus T_n^\alpha \subset \{\tau_j^{(i)} : i = 0, 1, 2, \ldots, \; j = 1, 2, \ldots, n\}. \tag{2.23}$$

Thus for each deterministic N we have

$$\bigcap_{n \geq N} (\bar{T}_n^\alpha \setminus T_n^\alpha) = \emptyset \quad \text{almost surely.}$$

However, since the T_n^α are nested,

$$\Big(\bigcap_{n \geq 1} \bar{T}_n^\alpha\Big) \setminus \Big(\bigcap_{n \geq 1} T_n^\alpha\Big) \subset \bigcup_{N \geq 1} \bigcap_{n \geq N} (\bar{T}_n^\alpha \setminus T_n^\alpha)$$

so the left-hand side is also empty almost surely, as required. □

Finally, Lemma 2.10 is a standard application of the ergodic theorem but it is slightly lengthy to prove due to the setup required. We shall state the definitions used here, but the interested reader should see Chapter 24 in [9] for further details. First let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma-$finite measure space with $\theta : \Omega \to \Omega$ measurable, and let $(\mathbb{R}, \mathcal{B}, \nu)$ denote the Borel σ-algebra on $\mathbb{R}$ with the Lebesgue measure. $A \in \mathcal{F}$ is called invariant (under θ) if $\theta^{-1}(A) = A$, and a measurable function $f : \Omega \to \mathbb{R}$ is called invariant (under θ) if $f = f \circ \theta$. The map θ is called measure-preserving if

$$\mu(\theta^{-1}(A)) = \mu(A) \text{ for all } A \in \mathcal{F}$$

and such a θ is called ergodic if $\mathcal{F}_\theta := \{A \in \mathcal{F} \;:\; \theta^{-1}(A) = A\}$ (which is a $\sigma-$algebra) contains only sets of measure 0 and their complements.

The version of the ergodic theorem we shall use is a specific case of [9, Theorem 24.1]:

Theorem 2.22. *If $\theta : \Omega \to \Omega$ is ergodic and $f : \Omega \to \mathbb{R}$ is invariant, then $f = c$ almost everywhere, where $c \in \mathbb{R}$ is some constant.*

Proof of Lemma 2.10. Before we apply the ergodic theorem, we should formally construct our probability space. For each $i \in \{0, 1, 2, \ldots\}$ and $j \in \mathbb{N}$, take a Rademacher random variable $B_j^{(i)}$ and an exponential random variable $E_j^{(i)}$ of parameter 1. We view our space Ω as the set of sequences $(((B_j^{(i)}, E_j^{(i)})_{i \geq 0})_{j \geq 1})$, i.e. $((\{-1, 1\} \times \mathbb{R}_+)^{\mathbb{N}_0})^{\mathbb{N}}$ with the product σ-algebra $\mathcal{F}$ and product measure μ. As usual for product σ-algebras, $\mathcal{F}$

can be defined as the smallest σ-algebra such that all the projection maps

$$\pi^{\omega}_{i,j} : \Omega \to \mathbb{R},\ (((B^{(i)}_j, E^{(i)}_j)_{i\geq 0})_{j\geq 1}) \mapsto \omega^{(i)}_j$$

are measurable; where $\omega \in \{B, E\}$, $i \in \mathbb{N}_0$, $j \in \mathbb{N}$. Equivalently $\mathcal{F}$ can be derived using cylinder sets, these are sets of the form

$$\bigotimes_{i,j,\omega} A^{\omega}_{i,j}$$

where again $i \in \mathbb{N}_0$, $j \in \mathbb{N}$ and $\omega \in \{B, E\}$ where each set of the form $A^{E}_{i,j}$ is an element of $\mathcal{B}(\mathbb{R}_+)$ but all but finitely many of them (over i, j) are equal to $\mathbb{R}_+$, while the sets of the form $A^{B}_{i,j}$ are each elements of $\mathcal{P}(\{-1, 1\})$ and all but finitely many of them (over i, j) are equal to $\{-1, 1\}$. If we call this collection of cylinder sets $\mathcal{C}$, then $\mathcal{C}$ is a π-system and $\sigma(\mathcal{C}) = \mathcal{F}$ as desired (see Chapter 2 of [31]).

We can now define $X_j(t)$ to take the value $B^{(i)}_j$ whenever $\sum_{k<i} E^{(i)}_j \leq t < \sum_{k\leq i} E^{(i)}_j$. Define the shift map $\theta : \Omega \to \Omega$, $(((B^{(i)}_j, E^{(i)}_j)_{i\geq 0})_{j\geq 1}) \mapsto (((B^{(i)}_j, E^{(i)}_j)_{i\geq 0})_{j\geq 2})$; in practical terms, θ deletes $X_1(t)$ and builds our (dynamical) random walk from the sequence $(X_2(t), X_3(t), \ldots)$ instead.

We show that θ is ergodic. We can restrict our attention to $\mathcal{C}$ as it generates $\mathcal{F}$. The preimage under θ of any cylinder set A is the same cylinder set shifted right, i.e. $\theta^{-1}(A) = (\{-1, 1\} \times \mathbb{R}_+) \times A$ which occurs with the same probability as A, hence θ is measurable and measure preserving (see e.g. [9, Lemma 24.1]). For ergodicity, define the tail σ-algebra $\mathcal{T}$ by

$$\mathcal{T} = \bigcap_{n\in\mathbb{N}} \mathcal{T}_n \text{ where } \mathcal{T}_n = \sigma(\pi^{\omega}_{i,j}\ :\ i \in \mathbb{N}_0,\ j > n,\ \omega \in \{B, E\}) \subset \mathcal{F}.$$

Now for $A \in \mathcal{C}$ (a cylinder set), we have that

$$\theta^{-n}(A) = \{x \in \Omega\ :\ \pi^{\omega}_{i,j+n}(x) \in A^{\omega}_{i,j}\ \forall i, j, \omega\} \in \mathcal{T}_n.$$

Now as $\mathcal{T}_n$ is a σ-algebra and $\mathcal{C}$ is a π-system, we have that $\theta^{-n}(A) \in \mathcal{T}_n\ \forall A \in \mathcal{F}$. Now if $A \in \mathcal{F}_\theta = \{A \in \mathcal{F}\ :\ \theta^{-1}(A) = A\}$ (i.e. a shift invariant set), then $A = \theta^{-n}(A) \in \mathcal{T}_n$ for every n, so $A \in \mathcal{T}$, this immediately gives us that $\mathcal{F}_\theta \subset \mathcal{T}$, and so by Kolmogorov's $0-1$ law (see e.g. [5, Theorem 7.2.4]) we have that $\mathcal{T}$ is trivial, hence $\mathcal{F}_\theta$ is trivial, so θ is ergodic. Define

$$\mathcal{E}'_\alpha = \Big\{t \in [0, 1] : \liminf_{n\to\infty} \frac{-Z_n(t)}{n^\alpha} > 0\Big\}.$$

For any $\alpha \geq 0$, the Hausdorff dimension of $\mathcal{E}_\alpha \cup \mathcal{E}'_\alpha$ is invariant under θ (as removing $X_1(t)$ will not affect whether or not the walk diverges), and therefore constant almost surely by Theorem 2.22. By symmetry, the Hausdorff dimension of $\mathcal{E}_\alpha$ equals that of $\mathcal{E}'_\alpha$. Since the Hausdorff dimension of the union of two sets is the maximum of their Hausdorff dimensions, the Hausdorff dimension of $\mathcal{E}_\alpha$ must therefore equal that of $\mathcal{E}_\alpha \cup \mathcal{E}'_\alpha$, and thus be constant almost surely. $\square$

Chapter 3

Exceptional Times Of Dynamical Brownian Motion

In this chapter, we extend the results from the previous chapter into continuous space. That is, we define a process called dynamical Brownian motion, which over time, evolves as a Brownian motion with reflections taking place in different locations.

In this chapter we prove results echoing Theorems 2.1 and 2.2 for our new model. As we will see, moving into continuous space makes certain aspects of our proof easier, and others significantly more challenging.

While many of the proofs are very similar to work done in the previous chapter, which was work done jointly with Matthew Roberts in [40], most proofs are still included here in full detail so that this chapter is as self-contained as possible. However, there will be explicit mentions to the previous chapter.

3.1 Introduction and results

A general treatment of Brownian motion can be found in [35] but we shall state the basics here. A one dimensional Brownian motion $B = (B_s)_{s\geq 0}$ started from $x \in \mathbb{R}$ satisfies:

- $B_0 = x$.
- B has independent increments, that is, if $s_2 > s_1$ then $B_{s_2} - B_{s_1}$ is independent of B_s for $s \leq s_1$.

- For any $0 \leq s_1 \leq s_2$, $B_{s_2} - B_{s_1} \sim N(0, s_2 - s_1)$ - a normal random variable with mean 0 and variance $s_2 - s_1$.

- B_s is continuous in s.

A Brownian motion can also be viewed as a scaling limit of a random walk via Donsker's Invariance Principle. That is, for any one-dimensional simple symmetric random walk (e.g. the switch walk $(Z_n)_{n\geq 0}$) we have:

$$\left(\frac{Z_{\lfloor ns \rfloor}}{\sqrt{n}}\right)_{s\in[0,1]} \to (B_s)_{s\in[0,1]}$$

in distribution (in Skorohod space, to be exact).

In order to relate the previous chapter to Brownian motion, we need to somehow implement similar dynamics into it, i.e. we need Brownian analogues to the "compass" and "switch" random walks.

The main issue with defining the process is that we can't think about rerandomising "bits" anymore, since a Brownian motion is a continuous path rather than a discrete number of steps. However we can fix this as by using a two dimensional Poisson point process (PPP) rather than independent one dimensional PPPs on each step as in the random walk case.

Recall that a Poisson point process (PPP) on $\mathbb{R}_+^2 := [0, \infty)^2$ of rate $\lambda > 0$ satisfies the following:

- The number of points in $A \subset \mathbb{R}_+^2$, denoted $N(A)$, is a $Poisson(\lambda|A|)$ random variable, where $|A|$ denotes the (Lebesgue) area of A. That is;

$$\mathbb{P}(N(A) = n) = \frac{(\lambda|A|)^n}{n!} e^{-\lambda|A|} \ \forall n \in \mathbb{N}_0.$$

- The number of points in disjoint subsets of $\mathbb{R}_+^2$ are independent.

The process we will be using is $\mathcal{P}$, a PPP on $\mathbb{R}_+^2$ of rate one where additionally; every point is kept with probability 1/2 and discarded otherwise. By the colouring/thinning theorem [27, Proposition 5.5 and Corollary 5.9] $\mathcal{P}$ is equivalent to a PPP(1/2).

This may seem like an unnecessary step as in fact a PPP of any constant non-zero rate will do, but this thinning process is meant to emulate the discrete case, where our rerandomisations only cause an actual change with probability 1/2.

A point in $\mathcal{P}$ is denoted (s,t), we refer to s as "Brownian time" and t as "dynamical time". This may seem confusing but it is standard in literature to say that B_s is a Brownian motion at time s, but as we will be considering $B_s(t)$ shortly the distinction becomes helpful. In the random walk case, we always wrote "steps" to refer to (discrete) "random walk time".

At times we will be focusing on a subset of $\mathbb{R}_+^2$ so we define for $A, B \subset \mathbb{R}_+$ (which for our work only need to be connected, i.e. intervals)

$$R_{A,B} := \{(s,t) \in \mathcal{P} \mid s \in A,\ t \in B\}$$

which can be viewed as a PPP of rate $1/2$ on the subset $A \times B$ of our ambient space $\mathbb{R}_+^2$. For $a, b \in \mathbb{R}_+$ we use the shorthand $R_{a,b}$ for $R_{[0,a),[0,b)}$. It doesn't matter whether or not we have closed or half-open intervals in the above definition, as the probability we have a point in e.g. $\mathbb{R}_+ \times \{t\}$ is zero as it has no Lebesgue-area.

Before defining dynamical Brownian motion, we introduce the following definition. For $t \geq 0$ set $I_0(t) = 0$, and for $k \geq 1$ define

$$I_k(t) = \min\{s > I_{k-1}(t) \mid \exists t' \leq t \text{ s.t. } (s,t') \in R_{\infty,t}\}.$$

That is, $I_k(t)$ is the Brownian time where the kth point (with respect to Brownian time) before dynamical time t of $\mathcal{P}$ is. This is the continuous time analogue of $I_k(t)$ from Chapter 2.

We now define dynamical Brownian motion as follows. Start with $\mathcal{P}$ and a Brownian motion $(B_s)_{s\geq 0}$ started from 0, and define $B_s(0) = B_s$ $\forall s$. For each $t > 0$, list the points of $R_{\infty,t}$ such that the first coordinates are in ascending order, that is, $(I_1(t), t_1(t)), (I_2(t), t_2(t)), \ldots$. Now, we define for all $s \in [I_j(t), I_{j+1}(t))$

$$B_s(t) := 2\sum_{i=1}^{j}(-1)^{i-1}B_{I_i(t)} + (-1)^j B_s.$$

This process allows us to define the dynamical Brownian motion $B = (B_s(t))_{s,t\geq 0}$ and we write $B(t) = (B_s(t))_{s\geq 0}$. Note that this construction is valid for any continuous function, in particular Brownian motions started from values other than one (e.g. the process $(\mathcal{B}_s)$, to be defined later). This formulation comes from the perspective of fixing dynamical time and moving through Brownian time, but it is also natural to fix Brownian time and then move through dynamical time. This can be done as follows.

Again, start with a Brownian motion path at dynamical time 0, $B(0) := (B_s(0))_{s\geq 0}$, started from 0, and the PPP $\mathcal{P}$. First we define $B_0(t) = B_0(0)\ \forall t > 0$, then to find $B_s(t)$ for $s, t \geq 0$:

1. List the points in $R_{s,t}$ in ascending order in the dynamical-time coordinate; $(s_1, t_1),\ \ldots, (s_n, t_n)$ for $t_1 < \cdots < t_n$ (a.s. there are finitely many points in $R_{s,t}$, and none of them share either coordinate, so the following is well defined).

2. Create a new Brownian path by reflecting $B(0)$ from Brownian time s_1 onwards by the horizontal line of height $B_{s_1}(0)$

3. Take this new path and reflect it from Brownian time s_2 onwards by the value the path has at Brownian time s_2 to create a new path.

4. Repeat this for $s_3, s_4, \ldots, s_n$ and the final path you have is $(B_u(t))_{u\leq s}$.

Note that as reflections commute we can actually do the reflections in the construction in any order provided you reflect at the relevant point. That is, the value of the Brownian motion at that Brownian time given the reflections you've already implemented. This means that both constructions given above are equivalent, albeit the former is nicer in that it gives an explicit formulation with far less effort. This model is indeed the continuous version of the switch random walk, as a single change in that model corresponded to a reflection of infinite length (if no other changes), just like in this model.

We do not define a dynamical Brownian motion to mirror the compass random walk because a single change in that model corresponded to a reflection of length one, and viewing Brownian motion as a limit of scaled discrete steps, a reflection of length one becomes a reflection of length $\delta > 0$ for arbitrarily small δ, which would not change the sample path. However it may be possible to make up for the effect of a single change being small by having more changes occur per unit time. We do not investigate this here but see [38] for reference.

As reflected Brownian motions are Brownian motions, it is clear that $\forall t\ B(t) = B(0)$ in distribution, so as in the random walk case the dynamics do not alter the distribution of the process at a fixed dynamical time. Thus we can ask the same questions of our dynamical Brownian motion as we did for the dynamical switch random walk.

We state the main theorem we prove in this chapter, which is an analogue of Theorem

2.1. Define for $\alpha \geq 0$ and $(B_s(t))$ the dynamical Brownian motion started from 0,

$$\mathcal{E} = \Big\{ t \in [0,1] : B_s(t) \to \infty \text{ as } s \to \infty \Big\}$$
$$\mathcal{E}_\alpha = \Big\{ t \in [0,1] : \liminf_{s\to\infty} \frac{B_s(t)}{s^\alpha} > 0 \Big\}.$$

Theorem 3.1. *There exist exceptional times of transience for Brownian motion: $\mathcal{E}$ is non-empty almost surely. In fact, the Hausdorff dimension of $\mathcal{E}_\alpha$ is $1/2$ almost surely for any $\alpha \in [0, 1/2)$. On the other hand, $\mathcal{E}_\alpha$ is empty almost surely for any $\alpha > 1/2$.*

We also have a noise sensitivity result to analogue Theorem 2.2

Theorem 3.2. *Let $(\varepsilon_s, s > 0)$ be any sequence in $(0,1)$ such that $s\varepsilon_s \to \infty$. The sequence of events $(\{B_s > 0\}, s > 0)$ is quantitatively noise sensitive with respect to the sequence $(\varepsilon_s, s > 0)$, by which we mean that*

$$\mathbb{P}(B_s(0) > 0 \text{ and } B_s(\varepsilon_s) > 0) - \mathbb{P}(B_s(0) > 0)^2 \to 0$$

as $s \to \infty$.

3.2 Structure and preliminaries

3.2.1 Layout of chapter

It is highly recommended that the reader who has read the previous chapter also reads Sections 3.2.2 and 3.2.3. This is because many elements of the proofs in this chapter echo equivalent proofs in the previous chapter, and these sections communicate the main differences between the two chapters. This means that a reader well-versed in Chapter 2 can skim through parts of this chapter that they are comfortable with and instead focus on the ideas in this chapter that are novel.

The rest of this chapter is organised as follows. In Section 3.2.4 we outline some useful facts about Brownian motions and PPPs that will be used extensively in our proofs. We prove Theorem 3.2 in Section 3.3. As in the previous chapter, the proof of Theorem 3.1 is significantly longer than Theorem 3.2, and so we give a proof outline in Section 3.4. In this outline we see that we must prove Proposition 3.12 for the lower bound on the Hausdorff dimension, which is done in Section 3.5, and Proposition 3.17 for the upper bound, proved in Section 3.6. In Section 3.7 we prove technical lemmas regarding the closure of the set of exceptional times and ergodicity that are required in the proof of Theorem 3.1. Finally, Sections 3.8 and 3.9 deal with technical results that are novel (relative to Chapter 2) and are detailed explicitly in the following subsection.

3.2.2 Guide for the rest of the chapter

The correspondence between the majority of sections in Chapters 2 and 3 is as follows:

$$\text{Section } 3.a.b \longleftrightarrow \text{Section } 2.a.b \qquad \forall 3 \leq a \leq 7$$

The proofs in these sections end up being quite similar, and a lot of the changes are natural changes one would expect when moving from discrete to continuous space, see Section 3.2.3 for full details. In this chapter we do not have a section dedicated to sketch proofs as in Section 2.2.3 as the sketches are identical. All that needs to be done is replace the random walks with Brownian motions, and change a couple of definitions (see the following subsection). For the rest of this subsection, we discuss the additional subsections added to this chapter and why they are needed, as well as any other significant changes to the proofs.

We start with Section 3.8. Recall from the previous chapter that it was Lemma 2.11, which came from [44, Theorem 8.1], that gave us the upper bound for our Hausdorff dimension. These results utilise the concept of the "influence" of a Boolean function, that is, the probability that changing a given bit will change the output of the function. Changing a single bit does not make sense in continuous space, so we must make a new definition. Full details of these changes are given in Section 3.4.2, but in short, rather than considering a bit changing we consider the Brownian motion path where a reflection does (or does not) occur at a given Brownian time. In other words, whether or not $(s,t) \in \mathcal{P}$ for a given s and any t. We then change the definition of total influence to be

$$\int_0^\infty \mathcal{I}_m(A_n) \ \mathrm{d}m$$

rather than a sum over m. Of course we need to prove that these definitions allow us to prove a Brownian version of Lemma 2.11, and that is what Theorem 3.15 and Corollary 3.16 are for. These are proven in Section 3.8. It is interesting to note that we heavily use the fact we have a built in Poisson point process in these proofs. In particular, we take advantage of Mecke's equation [27, Theorem 4.1].

It is worth mentioning at this stage that the other difference between Brownian motions and random walks that poses great difficulty is that after n steps (or n Brownian time) a simple random walk is bounded while a Brownian motion is not. This makes concluding the proof of Theorem 3.1 in Section 3.4.4 more challenging. It also forces us to compute influences for a range of Brownian motions started from different (fixed) values in Section 3.6.

Turning to Section 3.9, where we prove that a Brownian motion started from one stays above $(s+1)^\alpha - 1$ for all Brownian time $s \leq n$ occurs with probability $\asymp n^{-1/2}$. For the Chapter 2 equivalent, Lemma 2.13, [42, Theorem 2] does most of the legwork for us, and we do not have the luxury of a similar result in the Brownian case. Therefore, we employ results from stochastic calculus, such as Girsanov's Theorem, as well as work with the class of martingales of the form

$$\exp\left(\int_0^t \theta_s \, \mathrm{d}B_s - \frac{1}{2}\int_0^t \theta_s^2 \, \mathrm{d}s\right).$$

Once these results have been established, the rest of the work just involves delicately working with bounds.

3.2.3 How moving into continuous space changes the proofs

In this subsection we highlight consistent differences between the proofs in Chapters 2 and 3 that the reader should be aware of.

The first key difference is that since we are moving from discrete space to continuous space, a lot of key events and random variables need their definitions adjusted. For example we must take supremums and infimums of Brownian motions rather than maximums and minimums. This also impacts our definitions, e.g. we have already seen

$$I_k(t) = \min\{s > I_{k-1}(t) \mid \exists t' \leq t \text{ s.t. } (s, t') \in R_{\infty,t}\}$$

the first Brownian time after $I_{k-1}(t)$ where some reflection has occurred at some point in the first t dynamical time.

The waiting time $J_k(t) = I_k(t) - I_{k-1}(t)$ is now $Exp(t/2)$ distributed as opposed to $Geom((1-e^{-t})/2)$. This isn't surprising given that the exponential distribution is the continuous version of the geometric distribution, in the sense that they both satisfy the memoryless property. Often, e.g. in the proof of Theorem 3.2, the bounds calculated will be identical just with t instead of $1 - e^{-t}$, or another minor adjustment if the variance of $J_k(t)$ is used as this takes a different form with exponential random variables compared with geometric random variables. This change also affects the computation of $\mathbb{P}(A_1'(t))$ (compare the proof of Proposition 2.17 to Proposition 3.22's proof).

Other definitions change as well. For example we will have for $j \geq 1$

$$A_j(t) = \{\mathcal{B}_s(0) > 0 \text{ and } \mathcal{B}_s(t) > 0 \ \ \forall s \in [I_{j-1}(t), I_j(t))\},$$

and for even j

$$A'_j(t) = \{B_i^{(j)}(t) \in (0, 2W_{I_{j-1}(t)}(t)) \ \ \forall i \in [0, J_j(t))\}.$$

where $(\mathcal{B}_s)_s$ is a Brownian motion started from 1 and both $B_i^{(j)}(t)$ and $W_{I_{j-1}(t)}(t)$ are the Brownian analogues to their respective quantities from the previous chapter. These differ compared to their random walk counterparts due to the how we can now use the half-open interval $[I_{j-1}(t), I_j(t))$ as opposed to $[I_{j-1}(t), I_j(t) - 1]$ and so on, as we are in continuous time rather than discrete time. This actually makes our lives easier as we do not have the hassle of dealing with "-1"'s constantly. This lack of -1 also presents itself when we attach Brownian motions (formerly random walks like $B_i^{(j)}(t)$) to our averaged process W as the line of reflection is precisely the height of the Brownian motion where the reflection occurs.

When we proved Lemma 2.14, we converted our geometric random variable into a binomial random variable and used a concentration inequality (i.e. Chernoff bound). We can do the same for our exponential random variables using the definition of a PPP, but we get Poisson random variables instead.

As mentioned earlier, we use a two dimensional PPP(1/2) process to determine when and where we reflect our process. Thus every point in $\mathcal{P}$ corresponds to a reflection. As mentioned in the previous chapter, we did not have reflection times in the discrete case, but rerandomisation times, so one might argue a PPP(1) should be used. This does not matter in the slightest as a PPP(λ) for any constant $\lambda > 0$ would work, all that changes is the constants that appear in our bounds. The reason a PPP(1/2) is preferable to a PPP(1) is because in virtually all of the discrete time arguments we use $I_k(t)$ which tracks changes, not rerandomisations, so PPP(1/2) allows the constants in this chapter and the previous to match up. The one exception to this is when we prove that there are no exceptional times when $\alpha > 1/2$ as in the discrete case we track rerandomisations, this causes section 3.4.3 to slightly differ from section 2.4.3.

Another big change is that we use $\mathcal{B}_s(t) = B_s(t) + 1$, a standard Brownian motion started from 1, rather than $B_s(t)$ which starts from 0 when dealing with events such as,

$$P_n^\alpha(t) = \{\mathcal{B}_s(t) > (s+1)^\alpha - 1 \ \forall\ 0 < s \leq n\}$$

for $n \in \mathbb{R}_+$ (note $\mathbb{R}_+$ rather than $\mathbb{N}$), where we write $P_n(t)$ for $P_n^0(t)$ and P_n^α for $P_n^\alpha(0)$.

This is because a Brownian motion started from x will hit x infinitely many times in $[0, \varepsilon)$ for any $\varepsilon > 0$, almost surely. Thus we need to make some space between the origin and our Brownian motion otherwise all of our events would be empty. This doesn't impact our main theorems as they are about limiting behavior which of course is unaffected by a translation of size one.

Also, notice that we use $(s+1)^\alpha - 1$ rather than s^α, which was used in the random walk case. This is as we now can consider Brownian times arbitrarily close to 0 rather than from the first step onwards, so to ensure that P_n^α is decreasing in α for all n we must shift the path by 1. This also ensures that the derivatives of the curve behave nicely over all $s \geq 0$, which makes computing $\mathbb{P}(P_n^\alpha)$ doable as it is one of the few computations that is significantly harder in the Brownian motion case. We deal with this in Section 3.9. This does not cause any complications with $\mathcal{E}_\alpha$ as $s^\alpha \sim (s+1)^\alpha - 1$.

In the random walk proofs, we needed the probability that a random walk stays within a tube for a certain number of steps, given its initial position somewhere inside that tube. We need an analogous result for Brownian motions and different machinery is required. We utilise the Feynman-Kac formula which solves the heat equation in one dimension via an expectation of a specific random variable, which is seen in [35, Theorem 7.43]. This theorem has a power series representation which we shall elaborate on when we prove Proposition 3.17.

When dealing with technicalities involving ergodicity, we have to set up a different measure space. We will still use the definitions provided in Section 2.7, but as we will see in Section 3.7, the proof does differ slightly.

Finally, in Chapter 2 we use the FKG inequality which is only defined for functions of a discrete number of variables, such as random walks. However extensions of FKG that work with continuous functions such as Brownian motions do exist and will be used instead. See the next subsection.

3.2.4 Preparatory results

Throughout, we write $f(n) \lesssim g(n)$ if there exists a constant $c \in (0, \infty)$ such that $f(n) \leq cg(n)$ for all large n, and $f(n) \asymp g(n)$ if both $f(n) \lesssim g(n)$ and $g(n) \lesssim f(n)$.

If there f and g are functions of multiple variables then we shall make it clear which variable is "large" and how the other variables relate to said variable.

We use $\approx$ only in heuristics to mean "is roughly equal to". We write $\mathbb{P}_x$ for the probability measure under which our Brownian motions begin from x, rather than 0.

Just as $\mathbb{R}^2$ can be projected onto $\mathbb{R}$, we can project $\mathbb{R}_+^2$ onto either axis via a projection map. We denote these maps π_S and π_T for projections onto the Brownian/dynamical time axes respectively. Equally you can think of these maps as chopping off the second/first coordinate of $(s,t) \in \mathbb{R}_+^2$ respectively. It is clear that both of these maps are measurable with respect to the usual product sigma-algebra.

Lemma 3.3. *If $|B| < \infty$ then $\pi_S(R_{A,B})$ is a PPP on A of rate $|B|/2$ while if $|A| < \infty$ then $\pi_T(R_{A,B})$ is a PPP on B of rate $|A|/2$.*

Proof. The mapping theorem ([27, Theorem 5.1]) proves this instantly, but we sketch an argument for when A and B are connected intervals as this is the case we require for our present work.

Let A and B be as above. W.L.O.G we can consider $R_{a,b}$ for $a, b \in \mathbb{R}$ rather than $R_{A,B}$ due to translation invariance. We have almost surely that any two points in $R_{a,b}$ do not share either coordinate, so the number of points in $\pi_S(R_{a,b})$ equals the number of points in $R_{a,b}$, which is $Poisson(ab/2)$ distributed.

Further, for disjoint subsets $C, D \subset [0,a)$ the points in them corresponds to the points in $C \times [0,t)$, $D \times [0,t)$ respectively, which are disjoint regions. Thus we can say that $\pi_S(R_{a,b})$ is a PPP on $[0,a)$ of rate $b/2 = |B|/2$.

Similarly $\pi_T(R_{a,b})$ is a PPP on $[0,b)$ of rate $a/2 = |A|/2$. $\square$

In Chapter 2 we required the FKG inequality in order to produce particular bounds. This inequality is only defined for functions whose domains are discrete, so for our purposes we require an analogue that can work on Brownian paths. Indeed this work has been done in [6] for Brownian motions defined on $[0,T]$ for fixed $T \in \mathbb{R}$, and extended to Brownian motions on $\mathbb{R}^+$ in [34]. We detail the main things we need from these papers.

To make the setting explicity, denote $\Omega = C_0(\mathbb{R}^+)$ as the set of continuous real functions with domain $\mathbb{R}_+$ started from zero. Now let $(\Omega, \mathcal{F}, \mathbb{P})$ be the classical Wiener space for continuous processes started from zero. Next we require a partial-order on our set of Brownian paths in $\Omega = C_0(\mathbb{R}^+)$ so that "increasing" and "decreasing" make sense as concepts. For two paths $\omega_1, \omega_2 \in \Omega$, we say that $\omega_1 \leq \omega_2$ iff $\omega_1(t) \leq \omega_2(t)$ for all $t > 0$. In other words, the second path is always higher than the first.

We say that a function $f : \Omega \to \mathbb{R}$ is increasing if whenever $\omega_1 \leq \omega_2$ we have that $f(\omega_1) \leq f(\omega_2)$. We say f is decreasing iff $-f$ is increasing. An event $A \in \mathcal{F}$ (our σ-algebra) is increasing if whenever $\omega_1 \leq \omega_2$ we have that $\omega_1 \in A \implies \omega_2 \in A$, while A is decreasing if $\omega_1 \notin A \implies \omega_2 \notin A$. Equivalently, A is increasing/decreasing iff $\mathbb{1}_A$ is increasing/decreasing.

We now state the FKG inequality from [6] and [34]:

Proposition 3.4. *If $f, g : \Omega \to \mathbb{R}$ are measurable and increasing then*

$$\mathbb{E}[fg] \geq \mathbb{E}[f]\mathbb{E}[g].$$

In particular, if A, B are increasing events then

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B).$$

We remark that both [6] and [34] start their Brownian motions from zero when considering increasing events. However, we will be using the FKG inequality with Brownian motions started from $x > 0$. This is fine since there is translation-invariance present. To be explicit, we mean A is an increasing event for a Brownian motion started from zero iff $A + x$ (shift all sample paths satisfying A upwards by x) is an increasing event for Brownian motions started from x.

We now state and prove a corollary that is standard in the discrete setting, and indeed the proof is the same, but we include it for the sake of completeness:

Corollary 3.5. *If A is an increasing event and B is decreasing, then*

$$\mathbb{P}(A \cap B) \leq \mathbb{P}(A)\mathbb{P}(B).$$

Proof. Take $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \leq \omega_2$. B being decreasing means that if $\omega_1 \notin B$ then $\omega_2 \notin B$. Taking complements it is clear to see that B^c must be an increasing event. Using this alongside Proposition 3.4 gives

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}(A \cap B^c) \leq \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B^c) = \mathbb{P}(A)\mathbb{P}(B). \qquad \square$$

Let $x > 0$ be a constant. We have the following standard lemma that can be found in e.g. [10]:

Lemma 3.6. *For $y \le x$ we have*

$$\mathbb{P}_x(\inf_{s\le m} B_s \le y,\ B_m \in \mathrm{d}z) = \frac{1}{\sqrt{2\pi m}} \exp\left(-\frac{1}{2m}(|z-y| - y + x)^2\right) \mathrm{d}z.$$

We shall use Lemma 3.6 in the following way:

Corollary 3.7. *For $x, z \ge 0$ we have*

$$\mathbb{P}_x(\inf_{s\le m} B_s > 0,\ B_m \in \mathrm{d}z) = \frac{2}{\sqrt{2\pi m}} \exp\left(-\frac{z^2+x^2}{2m}\right) \sinh\left(\frac{zx}{m}\right) \mathrm{d}z.$$

Proof. We utilise Lemma 3.6 with $y = 0$ and obtain

$$\begin{aligned}\mathbb{P}_x(\inf_{s\le m} B_s > 0,\ B_m \in \mathrm{d}z) &= \mathbb{P}_x(B_m \in \mathrm{d}z) - \mathbb{P}_x(\inf_{s\le m} B_s \le 0,\ B_m \in \mathrm{d}z)\\ &= \frac{1}{\sqrt{2\pi m}}\left(\exp(-\frac{1}{2m}(z-x)^2) - \exp(-\frac{1}{2m}(z+x)^2)\right) \mathrm{d}z\\ &= \frac{2}{\sqrt{2\pi m}} \exp\left(-\frac{z^2+x^2}{2m}\right) \sinh\left(\frac{zx}{m}\right) \mathrm{d}z. \qquad \square\end{aligned}$$

We will want bounds on this probability, so we prove the following fact:

Lemma 3.8. *For $x \ge 0$ we have*

$$x \le \sinh(x) \le x \exp(x^2/6).$$

Proof. We simply look at the power series representation of sinh and note that $(2n+1)! \ge 6^n n!$ for all $n \ge 0$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \le x \sum_{n=0}^{\infty} \frac{x^{2n}}{6^n n!} = x e^{\frac{x^2}{6}}.$$

As $x \ge 0$ each term in the series is non-negative, and the first term is x, so the lower bound comes out instantly. $\square$

We next prove a Chernoff bound result that is a continuous analogue to Lemma 2.4:

Lemma 3.9. *For any $x > 0$ we have*

$$\mathbb{P}_0(B_1 \ge x) \le e^{-\frac{1}{2}x^2}.$$

Proof. Note that $B_1 \sim N(0,1)$ and thus has moment generating function $\exp(u^2/2)$. For any $u > 0$ we use Markov's inequality to obtain

$$\mathbb{P}_0(B_1 \geq x) = \mathbb{P}_0(e^{uB_1} \geq e^{ux}) \leq e^{-ux}\mathbb{E}[e^{uB_1}] = e^{-u\left(x-\frac{u}{2}\right)}.$$

Elementary calculus shows that the exponent is minimised when $u = x$, giving the required upper bound. □

We also utilise the reflection principle for Brownian motions, in [35, Theorem 2.16], which gives us a useful Lemma about the supremum and infimum of Brownian motions.

Lemma 3.10. *For any Brownian motion started from 0, we have that for all $r > 0$, $\sup_{0\leq s\leq r} B_s = |B_r|$ in distribution and $\inf_{0\leq s\leq r} B_s = -|B_r|$ in distribution.*

Proof. As the negative of a Brownian motion is itself a Brownian motion,

$$\sup_{0\leq s\leq r} B_s = -\inf_{0\leq s\leq r} -B_s = -\inf_{0\leq s\leq r} B_s$$

where the latter equality is in distribution only. The above tells us that proving the supremum result gets us the infimum result for free. This result for the supremum comes from [35, Theorem 2.18]. □

Corollary 3.11. *Let $C > 0$. For $r, z > 0$ such that $z^2 \leq Cr$ (z may depend on r) we have for sufficiently large r that*

$$\mathbb{P}_0(\sup_{s\leq r} B_s < z) = \mathbb{P}_0(\inf_{s\leq r} B_s > -z) \asymp zr^{-1/2}.$$

In particular if z is constant it may be ignored in the right hand side.

Proof. The equality is trivial by Lemma 3.10. The same Lemma alongside Brownian scaling and symmetry gives us

$$\mathbb{P}_0(\inf_{s\leq r} B_s > -z) = \mathbb{P}_0(|B_r| < z) = \mathbb{P}_0(|B_1| < zr^{-1/2}) = 2\mathbb{P}_0(0 < B_1 < zr^{-1/2}).$$

As $B_1 \sim N(0,1)$ we get

$$2\mathbb{P}_0(0 < B_1 < zr^{-1/2}) = \frac{2}{\sqrt{2\pi}} \int_0^{zr^{-1/2}} e^{-\frac{x^2}{2}}\,\mathrm{d}x.$$

The integrand is maximised when $x = 0$ and minimised when $x = zr^{-1/2}$ so we have

$$\frac{2z}{\sqrt{2\pi r}}e^{-\frac{z^2}{2r}} \leq \mathbb{P}_0(\inf_{s\leq r} B_s > -z) \leq \frac{2z}{\sqrt{2\pi r}}.$$

Now $\exp(-z^2/(2r)) \geq C' > 0$ (C' is a constant) by assumption, concluding the proof.

$\square$

3.3 Proof of Theorem 3.2: Noise Sensitivity of $\{B_s > 0\}$

This proof is shorter than the proof of Theorem 2.2 as we only need two parts rather than three. This is because we previously started off with the compass walk Y and then changed over to the switch walk Z, but this is not needed here. The only other difference is that by Lemma 3.3, reflections enter as a PPP$(\varepsilon_s/2)$ so $I_k(\varepsilon_s) - I_{k-1}(\varepsilon_s)$ is now a $Exp(\varepsilon_s/2)$ rather than a $Geom((1-e^{-\varepsilon_s})/2)$ random variable. This leads to the definition of $K(s)$ and the resulting bounds having ε_s in rather than $(1-e^{-\varepsilon_s})/2$.

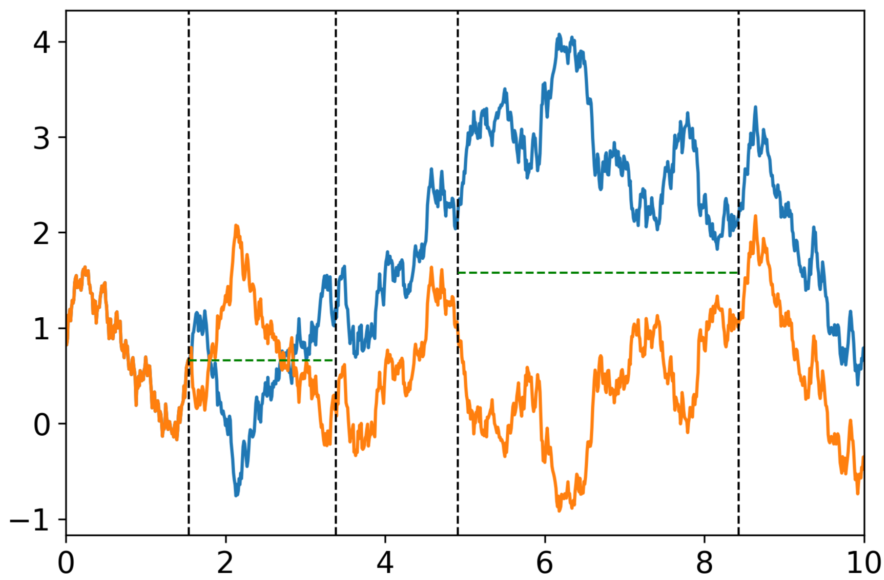

Figure 3-1: A realisation of $(\mathcal{B}_s(0))$ and $(\mathcal{B}_s(t))$ for $s \leq 10$. The black lines indicate the start of the next period. The green marks the line of symmetry between both paths on each even period.

Fix a sequence $(\varepsilon_s, s > 0)$ with $\varepsilon_s \in (0,1)$ for all s and $s\varepsilon_s \to \infty$. Many of the definitions in this section will depend implicitly on ε_s. Define for $t \geq 0$, $I_0(t) = 0$, and for $k \geq 1$,

$$I_k(t) = \min\{s' > I_{k-1}(t) \mid \exists t' \leq t \text{ s.t. } (s', t') \in R_{\infty,t}\}$$

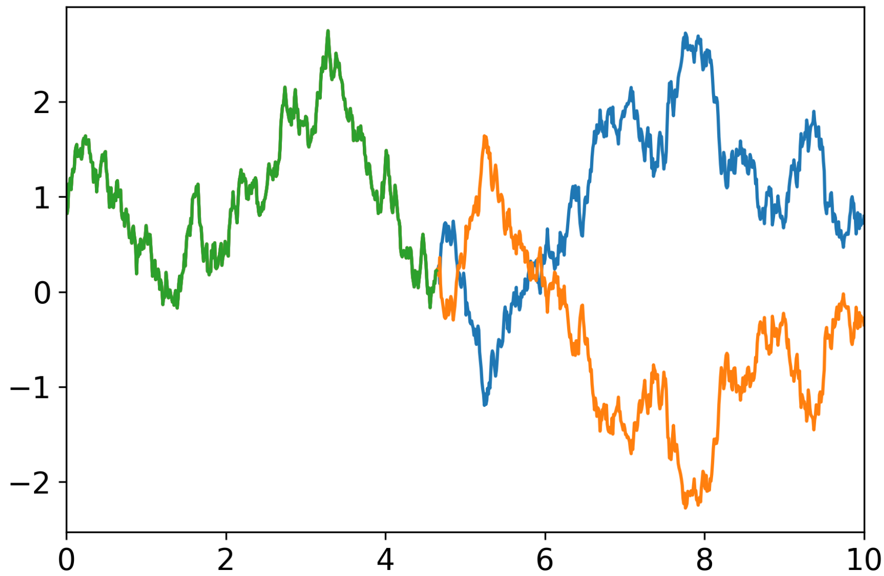

Figure 3-2: The two paths from Figure 3-1 where the odd and even periods have been cut and stuck together, the green path consists of all the odd periods (where the paths mirror each other).

the start of the $(k+1)$th period. The idea behind this proof is that on odd periods the $B(0)$ and $B(t)$ move in the same direction, while on even periods they mirror each other. Therefore we can cut and stick all the odd periods together, and all the even periods together. Then, for exactly one of the paths to end up above zero, we need the glued even path to have more height than the glued odd path, See Figures 3-1 and 3-2.

Let

$$K(s) = 2\lfloor s\varepsilon_s/4 \rfloor.$$

By the law of large numbers we have $I_{K(s)}(\varepsilon_s) \approx s$. We now prove Theorem 3.2 but with $I_{K(s)}(\varepsilon_s)$ in place of s. Then, we will transfer from using $I_{K(s)}(\varepsilon_s)$ to s.

Part 1: Proving Theorem 3.2 but with $I_{K(s)}(\varepsilon_s)$ in place of s.
Noting that $K(s)$ is even, define

$$U_s = \sum_{m=1}^{K(s)/2} B_{I_{2m-1}(\varepsilon_s)}(0) - B_{I_{2m-2}(\varepsilon_s)}(0)$$

and

$$V_s = \sum_{m=1}^{K(s)/2} B_{I_{2m}(\varepsilon_s)}(0) - B_{I_{2m-1}(\varepsilon_s)}(0).$$

In words, U_s is the sum of the increments of a Brownian motion over the odd periods up to Brownian time roughly s, and V_s is the sum over the even periods up to Brownian time roughly s. By this description, we clearly have that $B_{I_{K(s)}(\varepsilon_s)}(0) = U_s + V_s$. Moreover, since the increments of $B(\varepsilon_s)$ and $B(0)$ are equal on odd periods and mirrored on even periods, we have $B_{I_{K(s)}(\varepsilon_s)}(\varepsilon_s) = U_s - V_s$. Thus we have that

$$\mathbb{P}(B_{I_{K(s)}(\varepsilon_s)}(0) > 0 \text{ and } B_{I_{K(s)}(\varepsilon_s)}(\varepsilon_s) > 0) = \mathbb{P}(U_s + V_s > 0 \text{ and } U_s - V_s > 0).$$

Now as we are considering the Brownian motion up until $I_{K(s)}(\varepsilon_s)$ we have the same number of terms in both U_s and V_s and each term has the same distribution, so $U_s = V_s$ in distribution.

As $B_s(0) \sim N(0, s)$ we have that $\mathbb{P}(B_s(0) > 0)^2 = 1/4$, so a limit in s is not needed here, unlike in the random walk case. This means we must show that

$$\lim_{s\to\infty} \mathbb{P}(U_s + V_s > 0 \text{ and } U_s - V_s > 0) = \lim_{s\to\infty} \mathbb{P}(U_s > |V_s|) = 1/4.$$

To see this, we observe that

$$\mathbb{P}(U_s > |V_s|) = \frac{1}{2}\mathbb{P}(|U_s| > |V_s|) = \frac{1}{4},$$

which holds by symmetry of U_s as well as the fact that $|U_s|$ and $|V_s|$ are IID continuous random variables. Again this is slightly different to the random walk proof, as we did not need the limit here. We have now established the theorem with $I_{K(s)}(\varepsilon_s)$ in place of s. Just as in the random walk case, any value of $\varepsilon_s \in (0, 1)$ will work thus far. But it cannot be too small in order for the next part to work, as we need $K(s)$ to be large.

Part 2: Transferring from $I_{K(s)}(\varepsilon_s)$ to s.
We claim that

$$\mathbb{P}(B_s(0) > 0 \text{ and } B_s(\varepsilon_s) > 0) = \mathbb{P}(B_{I_{K(s)}(\varepsilon_s)}(0) > 0 \text{ and } B_{I_{K(s)}(\varepsilon_s)}(\varepsilon_s) > 0) + o(1). \tag{3.1}$$

We will use the elementary bounds, for any events A, B, A' and B',

$$\mathbb{P}(A \cap B) \le \mathbb{P}(A' \cap B') + \mathbb{P}(A \setminus A') + \mathbb{P}(B \setminus B')$$

and

$$\mathbb{P}(A \cap B) \ge \mathbb{P}(A' \cap B') - \mathbb{P}(A' \setminus A) - \mathbb{P}(B' \setminus B).$$

For the upper bound, using the first fact above,

$$\begin{aligned}\mathbb{P}(B_s(0) > 0 \text{ and } B_s(\varepsilon_s) > 0) \le \mathbb{P}(B_{I_{K(s)}(\varepsilon_s)}(0) > 0 \text{ and } B_{I_{K(s)}(\varepsilon_s)}(\varepsilon_s) > 0)\\ + \mathbb{P}(B_s(0) > 0 \text{ but } B_{I_{K(s)}(\varepsilon_s)}(0) \le 0)\\ + \mathbb{P}(B_s(\varepsilon_s) > 0 \text{ but } B_{I_{K(s)}(\varepsilon_s)}(\varepsilon_s) \le 0),\end{aligned}$$

and for the lower bound, using the second fact above,

$$\begin{aligned}\mathbb{P}(B_s(0) > 0 \text{ and } B_s(\varepsilon_s) > 0) \ge \mathbb{P}(B_{I_{K(s)}(\varepsilon_s)}(0) > 0 \text{ and } B_{I_{K(s)}(\varepsilon_s)}(\varepsilon_s) > 0)\\ - \mathbb{P}(B_s(0) \le 0 \text{ but } B_{I_{K(s)}(\varepsilon_s)}(0) > 0)\\ - \mathbb{P}(B_s(\varepsilon_s) \le 0 \text{ but } B_{I_{K(s)}(\varepsilon_s)}(\varepsilon_s) > 0).\end{aligned}$$

We will show that

$$\mathbb{P}(B_s(0) > 0 \text{ but } B_{I_{K(s)}(\varepsilon_s)}(0) \le 0) \to 0;$$

the three other similar terms can be dealt with similarly. To do this, we first note that for any $x_s, y_s > 0$,

$$\begin{aligned}\mathbb{P}\big(B_s(0) > 0 \text{ but } B_{I_{K(s)}(\varepsilon_s)}(0) \le 0\big) \le \mathbb{P}\big(|I_{K(s)}(\varepsilon_s) - s| > x_s\big) + \mathbb{P}\big(B_s(0) \in (0, y_s)\big)\\ + \mathbb{P}\Big(B_s(0) \ge y_s \text{ but } \inf_{j\in[s-x_s,s+x_s]} B_j(0) \le 0\Big). \quad (3.2)\end{aligned}$$

We first consider $\mathbb{P}(|I_{K(s)}(\varepsilon_s) - s| > x_s)$. We use Markov's inequality to see that

$$\mathbb{P}\big(|I_{K(s)}(\varepsilon_s) - s| > x_s\big) \le \frac{\mathbb{E}\big[|I_{K(s)}(\varepsilon_s) - s|^2\big]}{x_s^2},$$

and using the fact that $I_{K(s)}(\varepsilon_s)$ is a sum of $K(s)$ independent $exp(\varepsilon_s/2)$ random variables, we have

$$\begin{aligned}\mathbb{E}\big[|I_{K(s)}(\varepsilon_s) - s|^2\big] &= \mathrm{Var}(I_{K(s)}(\varepsilon_s)) + \mathbb{E}[I_{K(s)}(\varepsilon_s)]^2 - 2s\mathbb{E}[I_{K(s)}(\varepsilon_s)] + s^2\\ &= \frac{4K(s)}{\varepsilon_s^2} + \frac{4K(s)^2}{\varepsilon_s^2} - \frac{4sK(s)}{\varepsilon_s} + s^2.\end{aligned}$$

Recalling that $K(s) = 2\lfloor s\varepsilon_s/4\rfloor$, the above is at most

$$\frac{2s}{\varepsilon_s} + s^2 - \Big(\frac{8s}{\varepsilon_s}\Big)\Big(\frac{s\varepsilon_s}{4} - 1\Big) + s^2 = \frac{10s}{\varepsilon_s}.$$

Thus

$$\mathbb{P}\big(|I_{K(s)}(\varepsilon_s) - s| > x_s\big) \le \frac{10s}{x_s^2 \varepsilon_s}.$$

Choosing the value $x_s = s^{5/8}/\varepsilon_s^{3/8}$, we have

$$\mathbb{P}\big(|I_{K(s)}(\varepsilon_s) - s| > x_s\big) \le \frac{10}{s^{1/4}\varepsilon_s^{1/4}} \to 0 \tag{3.3}$$

by our assumption that $s\varepsilon_s \to \infty$.

We now move on to the second term on the right-hand side of (3.2). Choosing $y_s = s^{3/8}/\varepsilon_s^{1/8}$, since $(B_j(0), j \ge 0)$ is a standard Brownian motion from 0 (so that $B_s(0)/\sqrt{s} = N(0,1)$ in law) and $y_s \ll s^{1/2}$, we have

$$\mathbb{P}\big(B_s(0) \in (0, y_s)\big) = \mathbb{P}\big(N(0,1) \in (0, y_s s^{-1/2})\big) \to 0. \tag{3.4}$$

For the final term in (3.2), by the strong Markov property and Lemma 3.10, we have

$$\begin{aligned}
\mathbb{P}\Big(B_s(0) \ge y_s \text{ but } \inf_{j \in [s-x_s, s+x_s]} B_j(0) \le 0\Big) &\le \mathbb{P}_0\Big(\sup_{j\in[0,x_s]} B_j(0) \ge y_s\Big) \\
&\quad + \mathbb{P}_{y_s}\Big(\inf_{j\in[0,x_s]} B_j(0) \le 0\Big) \\
&= 2\mathbb{P}_0(|B_{x_s}| \ge y_s) \\
&= 4\mathbb{P}_0(N(0,1) \ge y_s x_s^{-1/2}).
\end{aligned}$$

Since $x_s = s^{5/8}/\varepsilon_s^{3/8} \ll s^{6/8}/\varepsilon_s^{2/8} = y_s^2$, the above must converge to zero as $s \to \infty$. Combining this with (3.3) and (3.4), we see from (3.2) that

$$\mathbb{P}\big(B_s(0) > 0 \text{ but } B_{I_{K(s)}(\varepsilon_s)}(0) \le 0\big) \to 0.$$

This, together with very similar bounds on the other three terms mentioned above, establishes (3.1). Combining this with Part 1 completes the proof of Theorem 3.2.

3.4 Outline of the proof of Theorem 3.1: Hausdorff dimension of exceptional times is $1/2$

We now outline the main steps for making a rigorous proof of the fact that the Hausdorff dimension of

$$\mathcal{E}_\alpha = \Big\{t \in [0,1] : \liminf_{s\to\infty} \frac{B_s(t)}{s^\alpha} > 0\Big\}$$

is 1/2 almost surely for any $\alpha \in [0, 1/2)$. Since $\mathcal{E}_\alpha \subset \mathcal{E}_0$ for any $\alpha \geq 0$, it suffices to give an upper bound on the dimension of $\mathcal{E}_0$ and a lower bound on the dimension of $\mathcal{E}_\alpha$ for $\alpha \in (0, 1/2)$. This also implies that $\mathcal{E}$, the set of times t where $B_s(t) \to \infty$, is non-empty almost surely and therefore that there exist exceptional times of transience. We will proceed by stating a series of results, whose proofs we delay until later sections.

3.4.1 Lower bound on Hausdorff dimension of $\mathcal{E}_\alpha$

We first set up all of the relevant objects. This is all very similar to the set up as done in section 2.4.1 but for Brownian motion. Let $B = (B_s)_{s \geq 0}$ denote the standard Brownian motion from zero, and $\mathcal{B}_s = B_s + 1 \; \forall s$ so that $\mathcal{B} = (\mathcal{B}_s)_{s \geq 0}$ is a standard BM from one. We define $B(t)$ and $\mathcal{B}(t)$ similarly. The construction of $\mathcal{B}(t)$ still makes sense even though it starts from one rather than zero.

We're interested in whether there exist times t where $B_s(t)$ stays above a curve that is tending to infinity as $s \to \infty$. It is vital that we use $\mathcal{B}$ rather than B since a Brownian motion started from zero will hit zero infinitely often in any time interval.

It is clear that if B stays above the curve, then so does $\mathcal{B}$, meaning that $\mathcal{E}_\alpha^B \leq \mathcal{E}_\alpha^{\mathcal{B}}$ where $\mathcal{E}_\alpha^B = \mathcal{E}_\alpha$ and

$$\mathcal{E}_\alpha^{\mathcal{B}} = \left\{ t \in [0,1] : \liminf_{s \to \infty} \frac{\mathcal{B}_s(t)}{s^\alpha} > 0 \right\}.$$

Also for $\alpha > 0$ note that

$$\frac{\mathcal{B}_s(t)}{s^\alpha} = \frac{B_s(t)}{s^\alpha} + \frac{1}{s^\alpha}$$

and both expressions have the same liminf, so $\mathcal{E}_\alpha^{\mathcal{B}} = \mathcal{E}_\alpha^B$ for $\alpha > 0$. As we only need a lower bound for all $\alpha \in (0, 1/2)$ (as $\mathcal{E}_\alpha \subset \mathcal{E}_0$) this is sufficient to allow us to consider $\mathcal{B}$ rather than B.

Recall that for $n \in \mathbb{R}_+$

$$\begin{aligned} P_n^\alpha(t) &= \{\mathcal{B}_s(t) > (s+1)^\alpha - 1 \; \forall \; 0 < s \leq n\} \\ T_n^\alpha &= \{t \in [0,1] \; : \; P_n^\alpha(t) \text{ holds}\} \end{aligned}$$

and that we write $P_n(t)$ for $P_n^0(t)$ and P_n^α for $P_n^\alpha(0)$.

We write $\bar{T}_n^\alpha$ for the closure of T_n^α and $T^\alpha = \bigcap_n T_n^\alpha$. Note here that we have an uncountable intersection unlike in the random walk case. That is,

$$\bigcap_{r\in\mathbb{R}_+} T_r^\alpha \text{ rather than } \bigcap_{n\in\mathbb{N}} T_n^\alpha$$

but by the Archimedean property (that $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ where $n > x$) and the fact that the intersection is decreasing, we have that both of these intersections are equal and so we do not need to worry about this technicality.

Finally we define, for $\gamma \in [0, 1)$,

$$\Phi_n^\alpha(\gamma) = \frac{1}{\mathbb{P}(P_n^\alpha)^2} \int_0^1 \int_0^1 \frac{\mathbb{1}_{P_n^\alpha(t_1)\cap P_n^\alpha(t_2)}}{|t_2 - t_1|^\gamma} \,\mathrm{d}t_1 \,\mathrm{d}t_2.$$

The above definitions reemphasise why we consider $\mathcal{B}$ rather than B. Brownian motion started from x will hit x infinitely many times in any time period $[0, \varepsilon)$ ($\varepsilon > 0$) almost surely. So if we defined this event in terms of B we'd get the empty set almost surely. We could instead ask the standard Brownian motion to stay above -1, but this causes needless complications when dealing with the average process and the reflections later on.

We need Brownian equivalents to Lemmas and Propositions 2.7, 2.8, 2.9 and 2.10 that we will prove later. We can in fact use Lemma 2.7 directly from the previous chapter as our particular choice of the event P_n^α does not matter in said proof. The other three results are not trivial so must be proven again for our particular Φ and P_n^α events, so we state them again:

Proposition 3.12. *For any $\alpha, \gamma \in [0, 1/2)$,*

$$\sup_n \mathbb{E}[\Phi_n^\alpha(\gamma)] < \infty.$$

Lemma 3.13. *For any $\alpha \geq 0$, we have*

$$\bigcap_{n=1}^\infty \bar{T}_n^\alpha = \bigcap_{n=1}^\infty T_n^\alpha$$

almost surely.

Lemma 3.14. *For each $\alpha \geq 0$, the Hausdorff dimension of $\mathcal{E}_\alpha$ is a constant (possibly depending on α) almost surely.*

We prove Proposition 3.12 in section 3.5 and prove both Lemmas in section 3.7.

3.4.2 Upper bound on Hausdorff dimension of $\mathcal{E}_0$

We want to use a result like Lemma 2.11 to prove that the Hausdorff dimension of the exceptional times also has an upper bound of 1/2. However this result relies on the process being made dynamical to be discrete, so that pivotal probabilities etc. can be defined. We therefore must define what a pivotal probability is for a continuous process, and therefore the influence.

Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is the classical Wiener space for continuous processes started from some fixed $x \geq 0$ and let $A \in \mathcal{F}$. Denote our Brownian motion by B and let $B^{(m)}$ be that same process, but reflected at B_m, so as if there was a sole Poisson point at $(m, t) \in \mathcal{P}$ for some $t > 0$. Then $\mathbb{1}_A$ can be viewed as a function of a particular Brownian motion and we can define the pivotal probability as

$$\mathcal{I}_m^x(A) := \mathbb{P}(\mathbb{1}_A(B) \neq \mathbb{1}_A(B^{(m)}) \mid B_0 = x).$$

In other words, the probability that the reflection changes whether or not A occurs. The total influence in the discrete case is defined to be the sum of all the pivotal probabilities. As we are now in the continuous case, we do the obvious thing and use an integral instead:

$$\mathcal{I}^x(A) := \int_0^\infty \mathcal{I}_m^x(A)\, \mathrm{d}m.$$

We now state a continuous analogue to [44, Theorem 8.1] from which Lemma 2.11 was based:

Theorem 3.15. *Let (A_n) be events on $C_1(\mathbb{R}_+)$ (continuous real functions with domain $\mathbb{R}_+$ started from one) where for each n, there exists $s_n < \infty$ such that A_n depends only on the path up to Brownian time s_n. Let $\mathbb{P}(A_n) \to 0$. Let $(B(t))_t$ be the dynamical process built using the $PPP(1/2)$ on $\mathbb{R}_+^2$, started from the stationary distribution. Let $T = \{t \in [0, 1] \;:\; B(t) \in \cap_{n=1}^{\infty} A_n\}$. Then if $\liminf_{n\to\infty} \mathcal{I}^1(A_n) = \infty$, then the Hausdorff dimension of T is bounded above by*

$$\liminf_{n\to\infty} \left(1 - \frac{\log \mathbb{P}(A_n)}{\log \mathcal{I}^1(A_n)}\right)^{-1} \quad a.s.$$

The hope is that this theorem can be easily adapted to work for any dynamical process built using a PPP that evolves over time, not just dynamical Brownian motion. This is because the proof doesn't use the fact that $B(t)$ is a Brownian motion at all.

Note that [44, Theorem 8.1] has an extra part stating that if $\liminf_n \mathcal{I}^1(A_n) < \infty$ then $T = \emptyset$ almost surely, but the proof of that is highly technical and we do not need such a result for our purposes.

For technical reasons we require the following generalisations of P_n and T_n. Define for fixed $k, l \in \mathbb{N}_0$:

$$\begin{aligned} P_{k,l,n} &:= \{\mathcal{B}_k \in [l, l+1);\ \mathcal{B}_s > 0\ \forall s \in [k, k+n]\}, \\ T'_{k,l} &:= \{t \in [0,1]\ :\ \mathcal{B}_k(t) \in [l, l+1);\ \mathcal{B}_s(t) > 0\ \forall s \geq k\}. \end{aligned}$$

We will apply the preceeding Theorem in order to prove:

Corollary 3.16. *For all fixed $k, l \in \mathbb{N}_0$, the Hausdorff dimension of $T'_{k,l}$ is at most*

$$\liminf_{n\to\infty} \left(1 - \frac{\log \mathbb{P}(P_{k,l,n})}{\log \mathcal{I}^1(P_{k,l,n})}\right)^{-1}.$$

We shall prove both of these results later in section 3.8. We also require the asymptotic behavior of the pivotal probabilities in order to use these previous results. We have the following Proposition which will be proved in section 3.6:

Proposition 3.17. *Fix $x, \varepsilon > 0$. For any real $0 \leq m \leq n - \varepsilon$ we have that*

$$\mathcal{I}^x_m(P_n) \asymp \frac{n-m}{n^{3/2}}.$$

The implicit constants may depend on x and ε. In particular, the constant involved in bounding $\mathcal{I}^x_m(P_n)$ from above is monotonically increasing in x.

3.4.3 $\mathcal{E}_\alpha$ is empty for $\alpha > 1/2$

The following argument echoes the proof in Section 2.4.3. The key difference is that we no longer need to split our argument into the cases $1 > \alpha > 1/2$ and $\alpha \geq 1$. This is because now our Chernoff bound for $\mathcal{L}^\alpha(2)$ is optimised by a λ value that does not converge to 0 (in n) for all $\alpha > 1/2$.

The final part of Theorem 3.1 says that $\mathcal{E}_\alpha$ is empty almost surely when $\alpha > 1/2$. The proof of this fact follows a fairly standard argument. For $\alpha, t \geq 0$ and $s > 0$ define the event $L^\alpha_s(t) = \{B_s(t) \geq s^\alpha\}$, and for $k \in \mathbb{N}$ let $\mathcal{L}^\alpha_s(k) = \int_0^k \mathbb{1}_{L^\alpha_s(t)}\, \mathrm{d}t$. It suffices to consider $n \in \mathbb{N}$ rather than $s > 0$ for reasons explained at the end of the proof. Note that

$$\mathbb{P}(\mathcal{L}_n^\alpha(1) > 0) \le \mathbb{P}(\mathcal{L}_n^\alpha(1) > 0)\frac{\mathbb{E}[\mathcal{L}_n^\alpha(2)]}{\mathbb{E}[\mathcal{L}_n^\alpha(2)\mathbb{1}_{\{\mathcal{L}_n^\alpha(1)>0\}}]} = \frac{\mathbb{E}[\mathcal{L}_n^\alpha(2)]}{\mathbb{E}[\mathcal{L}_n^\alpha(2) \,|\, \mathcal{L}_n^\alpha(1) > 0]}. \tag{3.5}$$

By Fubini's theorem, stationarity, and Brownian scaling,

$$\mathbb{E}[\mathcal{L}_n^\alpha(2)] = \int_0^2 \mathbb{P}(B_n(t) \ge n^\alpha)\,\mathrm{d}t = 2\mathbb{P}(B_n \ge n^\alpha) = 2\mathbb{P}(B_1 \ge n^{\alpha-1/2}).$$

So by Lemma 3.9 we have

$$\mathbb{E}[\mathcal{L}_n^\alpha(2)] \le 2\exp\left(-\frac{1}{2}n^{2\alpha-1}\right). \tag{3.6}$$

On the other hand, letting $T = \inf\{t \in [0,1] : B_n(t) \ge n^\alpha\}$, we have

$$\mathbb{E}[\mathcal{L}_n^\alpha(2) \,|\, \mathcal{L}_n^\alpha(1) > 0] \ge \mathbb{E}\Big[\int_T^{T+1} \mathbb{1}_{L_n^\alpha(t)}\,\mathrm{d}t \,\Big|\, \mathcal{L}_n^\alpha(1) > 0\Big].$$

Let $T' = \inf\{t \ge T : \exists(s,t) \in R_{n,\infty}\}$, the first dynamical-time after T where a reflection takes place during the first n Brownian-time. This definition is subtly different from what was done in Section 2.4.3, as then we waited for the next rerandomisation while now we don't have rerandomisations, just reflections. Now, provided $T < \infty$,

$$\int_T^{T+1} \mathbb{1}_{L_n^\alpha(t)}\,\mathrm{d}t \ge (T' - T) \wedge 1.$$

$\pi_T(R_{n,\infty})$, which is a PPP$(n/2)$ by Lemma 3.3, represents the dynamical times where reflections occur among Brownian times up to n. $T' - T$ is the waiting time between two such reflections, so is exponentially distributed with parameter $n/2$. Thus

$$\begin{aligned}\mathbb{E}\Big[\int_T^{T+1} \mathbb{1}_{L_n^\alpha(t)}\,\mathrm{d}t \,\Big|\, \mathcal{F}_T\Big] \ge \mathbb{E}[(T'-T) \wedge 1] &= \int_0^1 s \cdot \frac{n}{2}e^{-\frac{n}{2}s}\,\mathrm{d}s \\ &\ge \frac{1}{2}\int_0^{1/n} nse^{-\frac{n}{2}s}\,\mathrm{d}s \ge \frac{1}{4\sqrt{e}n},\end{aligned}$$

where we've used that $\exp(-ns/2) \ge \exp(-1/2)$ in the integration region. We have

$$\mathbb{E}[\mathcal{L}_n^\alpha(2) \,|\, \mathcal{L}_n^\alpha(1) > 0] \ge \frac{1}{4\sqrt{e}n}.$$

Combining this with (3.5) and (3.6), for any $\alpha > 0$ we have

$$\mathbb{P}(\mathcal{L}_n^\alpha(1) > 0) \leq 2\exp(-n^{2\alpha-1}/2) \cdot 4\sqrt{e}n.$$

For $\mathcal{E}_\alpha$ to be non-empty we must have a time $t \in [0,1]$ where there exists $N \in \mathbb{N}$ such that $L_n(t)$ occurs for all integer $n \geq N$. But when $\alpha > 1/2$, the Borel-Cantelli lemma gives that there exists $N \in \mathbb{N}$ such that $\mathcal{L}_n^\alpha = 0$ almost surely for all $n \geq N$. We now show that $\mathcal{L}_n^\alpha = 0$ implies $L_n^\alpha(t)$ does not occur for all $t \in [0,1]$.

If $L_n^\alpha(t)$ occurs for some $t \in [0,1)$ then this would force an interval of time of positive Lebesgue measure to exist where L_n^α occurs, as we would have to wait for the next rerandomisation time. This is clearly contradictory. We are then left to check that $L_n^\alpha(1)$ cannot occur infinitely often. By equality in law and Fubini's theorem, it can be seen that

$$\mathbb{P}(L_n^\alpha(1)) = \mathbb{P}(L_n^\alpha(0)) = \mathbb{E}[\mathcal{L}_n^\alpha(1)] \leq \frac{1}{2} \times (3.6) = \exp\left(-\frac{1}{2}n^{2\alpha-1}\right).$$

Again, Borel-Cantelli shows that we cannot have $L_n^\alpha(1)$ occuring infinitely often, concluding the proof.

3.4.4 Concluding the proof of Theorem 3.1

We now tie together the results from Sections 3.4.1, 3.4.2 and 3.4.3 to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. We showed in Section 2.5.3 that $\mathcal{E}_\alpha$ is empty almost surely for $\alpha > 1/2$, so it remains to show that the Hausdorff dimension of $\mathcal{E}_\alpha$ is 1/2 for any $\alpha \in [0, 1/2)$. As stated at the beginning of Section 2.5, it suffices to show that the Hausdorff dimension of $\mathcal{E}_\alpha$ is at least 1/2 for $\alpha > 0$ and the Hausdorff dimension of $\mathcal{E}_0$ is at most 1/2.

By Lemma 2.7 and Proposition 3.12, we know that for any $\alpha, \gamma \in [0, 1/2)$, the Hausdorff dimension of $\bigcap_n \bar{T}_n^\alpha$ is at least γ with strictly positive probability. By Lemma 3.13, the same holds for T^α, and since $T^\alpha \subset \mathcal{E}_\alpha$, the same holds for $\mathcal{E}_\alpha$. Lemma 3.14 then tells us that the Hausdorff dimension of $\mathcal{E}_\alpha$ must be at least 1/2 almost surely.

For the upper bound, we aim to use Corollary 3.16 alongside the following decomposition:

$$\mathcal{E}_0 = \{t \in [0,1] : \liminf_{s\to\infty} B_s(t) > 0\} = \bigcup_{k,l \in \mathbb{N}_0} T'_{k,l}.$$

In particular, a countable union of sets of Hausdorff dimension at most 1/2 almost surely, itself has Hausdorff dimension at most 1/2 almost surely. So it suffices to consider $T'_{k,l}$ for each possible value of k and l.

The first (and most general) case we consider is where $k, l \neq 0$. Recall that

$$P_{k,l,n} := \{\mathcal{B}_k \in [l, l+1);\ \mathcal{B}_s > 0\ \forall s \in [k, k+n]\}.$$

We have that (via the Markov property)

$$\begin{aligned}\mathbb{P}_1(P_{k,l,n}) &= \mathbb{P}_1(B_s > 0\ \forall s \in [k, k+n] \mid B_k \in [l, l+1))\mathbb{P}_1(B_k \in [l, l+1)) \\ &\leq \mathbb{P}_{l+1}(P_n)\mathbb{P}_1(B_k \in [l, l+1)).\end{aligned}$$

As l is a fixed constant we also know that $\mathbb{P}_{l+1}(P_n) \asymp (l+1)n^{-1/2}$ by Corollary 3.11.

For an interval $I \subset \mathbb{R}$ and an event A, we write

$$\mathcal{I}_m^I(A) := \sup_{x \in I} \mathcal{I}_m^x(A)$$

and $\mathbb{P}_I$ for the law of a Brownian motion started from the value of a $N(1, k)$ (k will be known by the context) random variable conditioned to be in I. Now for $k < m \leq k+n$, we know that we require $\{B_k \in [l, l+1)\}$ in order to be pivotal for $P_{k,l,n}$, so

$$\begin{aligned}\mathcal{I}_m^1(P_{k,l,n}) &= \mathbb{P}_1(B_k \in [l, l+1),\ m \text{ is pivotal for } P_{k,l,n}) \\ &= \mathbb{P}_1(B_k \in [l, l+1))\mathbb{P}_1(m \text{ is pivotal for } P_{k,l,n} \mid B_k \in [l, l+1)) \\ &= \mathbb{P}_1(B_k \in [l, l+1))\mathbb{P}_{[l,l+1)}(m \text{ is pivotal for } P_n) \\ &\leq \mathbb{P}_1(B_k \in [l, l+1))\mathcal{I}_{m-k}^{[l,l+1)}(P_n)\end{aligned}$$

where the penultimate equality is by the Markov property. Note that for $m > k+n$ the pivotal probability is zero. Now, fixing $\varepsilon > 0$ as in Proposition 3.17, we have

$$\begin{aligned}\mathcal{I}^1(P_{k,l,n}) = \int_0^\infty \mathcal{I}_m^1(P_{k,l,n})\,\mathrm{d}m &= \int_0^k \mathcal{I}_m^1(P_{k,l,n})\,\mathrm{d}m + \int_k^{k+n} \mathcal{I}_m^1(P_{k,l,n})\,\mathrm{d}m \\ &\leq k + \varepsilon + \mathbb{P}_1(B_k \in [l, l+1)) \int_0^{n-\varepsilon} \mathcal{I}_m^{[l,l+1)}(P_n)\,\mathrm{d}m \\ &= k + \varepsilon + \mathbb{P}_1(B_k \in [l, l+1)) \int_0^{n-\varepsilon} \sup_{x \in [l,l+1)} \mathcal{I}_m^x(P_n)\,\mathrm{d}m.\end{aligned}$$

We have shown in Proposition 3.17 that for any fixed $x, \varepsilon > 0$ $\exists C_{x,\varepsilon} > 0$ such that

$$\mathcal{I}_m^x(P_n) \le C_{x,\varepsilon} \frac{n-m}{n^{3/2}} \quad \forall m, n.$$

A detailed look into the proof of the upper bound of Proposition 3.17, which can be found in Section 3.6.3 shows that for sufficiently large n C_x is monotonically increasing in x, so that

$$\sup_{x \in [l,l+1)} C_{x,\varepsilon} \le C_{l+1,\varepsilon}$$

which means that

$$\begin{aligned}\mathcal{I}^1(P_{k,l,n}) &\lesssim k + \varepsilon + C_{l+1,\varepsilon} n^{-3/2} \mathbb{P}_1(B_k \in [l, l+1)) \int_0^{n-\varepsilon} (n-m)\, \mathrm{d}m \\ &\asymp k + \varepsilon + C_{l+1,\varepsilon} n^{1/2} \mathbb{P}_1(B_k \in [l, l+1)).\end{aligned}$$

We therefore have that there exist constants $c, c' \in (0, \infty)$ such that

$$\frac{-\log \mathbb{P}(P_{k,l,n})}{\log \mathcal{I}(P_{k,l,n})} \ge \frac{\frac{1}{2}\log n + \log(l+1) - \log c - \log \mathbb{P}_1(B_k \in [l, l+1))}{\frac{1}{2}\log n + \log c' + \log(C_{l+1,\varepsilon}\mathbb{P}_1(B_k \in [l, l+1)) + (k+\varepsilon)n^{-1/2})},$$

which converges to 1 as $n \to \infty$ for each fixed k, l and ε. From Corollary 3.16 we obtain that when $k, l \neq 0$ the Hausdorff dimension of $T'_{k,l}$ is almost surely at most $(1+1)^{-1} = 1/2$.

Next, if $k = 0$ then $T'_{k,l}$ is clearly the empty set unless $l = 1$ and in this case the previous argument above holds exactly.

Finally we must consider the case where $l = 0$ (and $k \neq 0$) which is different as we leave open the possibility for our Brownian motion to get incredibly close to 0. To get around this we write

$$T'_{k,0} = \bigcup_{r \in \mathbb{N}_0} T'_{k,0,r}$$

where

$$T'_{k,0,r} = \{t \in [0,1] \ : \ B_k \in [2^{-(r+1)}, 2^{-r}),\ B_s > 0 \ \forall s > k\}.$$

So it now suffices to show that each $T'_{k,0,r}$ has Hausdorff dimension at most 1/2 almost surely. As B_k is now bounded, we can repeat the earlier arguments just with $[2^{-(r+1)}, 2^{-r})$ rather than $[l, l+1)$ and the proof follows exactly the same. □

3.5 Proof of Proposition 3.12: bounding $\Phi_n^\alpha(\gamma)$ from above

By Fubini's theorem,

$$\begin{aligned}\mathbb{E}[\Phi_n^\alpha(\gamma)] &= \frac{1}{\mathbb{P}(P_n^\alpha)^2}\mathbb{E}\Big[\int_0^1\int_0^1 \frac{\mathbb{1}_{P_n^\alpha(s)\cap P_n^\alpha(t)}}{|t-s|^\gamma}\,\mathrm{d}s\ \mathrm{d}t\Big] \\ &= \frac{1}{\mathbb{P}(P_n^\alpha)^2}\int_0^1\int_0^1 \frac{\mathbb{P}(P_n^\alpha(s)\cap P_n^\alpha(t))}{|t-s|^\gamma}\,\mathrm{d}s\ \mathrm{d}t.\end{aligned}$$

By stationarity, this is bounded above by

$$\frac{2}{\mathbb{P}(P_n^\alpha)^2}\int_0^1 \frac{\mathbb{P}(P_n^\alpha(0)\cap P_n^\alpha(t))}{t^\gamma}\,\mathrm{d}t,$$

and since $P_n^\alpha(u)\subset P_n(u)$ for any $\alpha, u\geq 0$, this is at most

$$\frac{2}{\mathbb{P}(P_n^\alpha)^2}\int_0^1 \frac{\mathbb{P}(P_n(0)\cap P_n(t))}{t^\gamma}\,\mathrm{d}t.$$

The following lemma says that the probability of P_n^α is of the same order as the probability as P_n. We prove it in Section 3.9 as the proof is significantly more involved than our equivalent proof for random walks.

Lemma 3.18. *For any $\alpha<1/2$ and $n>0$,*

$$\mathbb{P}(P_n^\alpha)\asymp\frac{1}{\sqrt{n}}.$$

In particular this holds for $\alpha=0$.

We now want to bound $\mathbb{P}(P_n(0)\cap P_n(t))$. As discussed in e.g. Section 3.3, we have that on even periods the time 0 and time t Brownian motions are mirrored, so we require them both to be larger than 0. We again must establish definitions to help accurately quantify the dependencies between different periods. We recall first that $I_0(t)=0$ and for $j\geq 1$

$$I_j(t)=\min\{s>I_{j-1}(t):(s,t)\in R_{\infty,t}\},$$

the jth Brownian-time for which a reflection takes place during the 1st t dynamical-time (w.r.t time 0). We call the Brownian-time between $I_{j-1}(t)$ and $I_j(t)$ the "jth period", and let $J_j(t)=I_j(t)-I_{j-1}(t)$ be the length of the jth period.

For each $j \geq 1$, define the event

$$A_j(t) = \{\mathcal{B}_s(0) > 0 \text{ and } \mathcal{B}_s(t) > 0 \ \ \forall s \in [I_{j-1}(t), I_j(t))\},$$

which says that our dynamical Brownian motion is positive throughout the jth period at both time 0 and time t. In contrast to the random walk proof we can use a half open interval rather than a closed interval with a $I_j(t) - 1$ in. Another difference is that we no longer need to make $j = 1$ a special case, as our Brownian motion starts above zero. This trend continues throughout the proof. For each $s \geq 0$, let

$$W_s(t) = \frac{\mathcal{B}_s(0) + \mathcal{B}_s(t)}{2},$$

the average of $\mathcal{B}(0)$ and $\mathcal{B}(t)$. Note that, for each t, during odd periods the differences $W_{s_2}(t) - W_{s_1}(t)$ are equal to $\mathcal{B}_{s_2}(0) - \mathcal{B}_{s_1}(0)$ almost surely (over all valid $s_2 \geq s_1$); and during even periods, $W_s(t)$ is constant.

When j is odd, define the event

$$A'_j(t) = \{W_s(t) > 0 \ \ \forall i \in [I_{j-1}(t), I_j(t))\}$$

that $W(t)$ is positive throughout the jth period. Note that, since $W_s(t)$ is the average of $\mathcal{B}_s(0)$ and $\mathcal{B}_s(t)$, if both of these are positive, then so is $W_s(t)$. That is, if j is odd, then $A_j(t) \subset A'_j(t)$.

Making the same comparison when j is even would not be useful since W is constant. Instead, when j is even, let $B^{(j)}_s(t)$, $i \geq 0$ be an independent Brownian motion started from $W_{I_{j-1}(t)}(t)$ and define

$$A'_j(t) = \{B^{(j)}_i(t) \in (0, 2W_{I_{j-1}(t)}(t)) \ \ \forall i \in [0, J_j(t))\}.$$

See Figure 3-3 for a realisation of all of these processes. We pause briefly to note another difference between this proof and the one in the previous chapter. The processes $B^{(j)}$ are allowed to start precisely on the point where the reflection occurred (i.e. at the point in the PPP), while in the random walk case we had to start our random walks one step earlier at $W_{I_{j-1}(t)-1}$.

We need to rule out some unlikely events. Let

$$E^{\text{odd}}_n(t) = \{J_3(t) + J_5(t) + \ldots + J_{2\lfloor nt/8 \rfloor+1}(t) \geq n/8\},$$

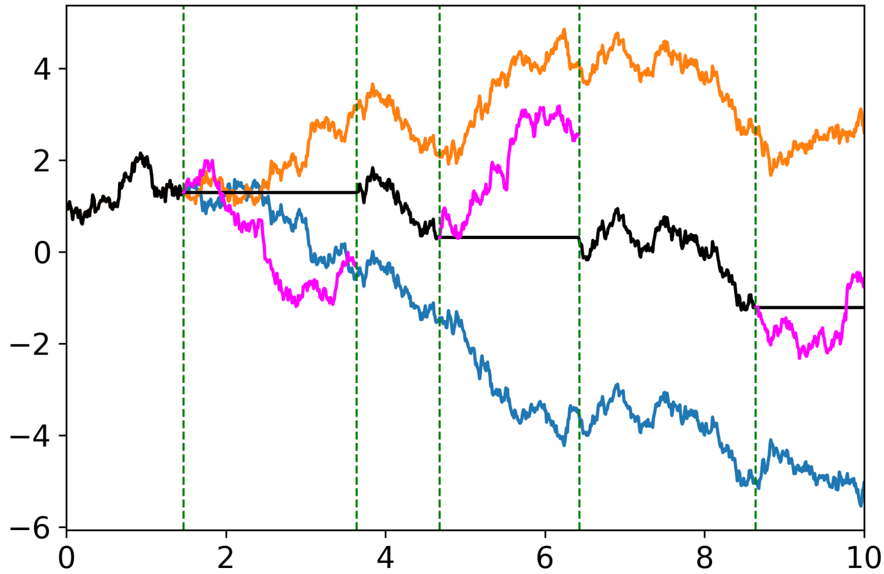

Figure 3-3: A realisation of $B(0)$ and $B(t)$ (blue/orange), $W(t)$ (black) and $B^{(2)}(t)$,$B^{(4)}(t)$ and $B^{(6)}(t)$ (magenta).

which we think of as the event that the odd periods (not including the first) are not too short,

$$E_n^{\text{even}}(t) = \{J_2(t) + J_4(t) + \ldots + J_{2\lfloor nt/8 \rfloor}(t) \geq n/8\},$$

which we think of as the event that the even periods are not too short,

$$E_n(t) = E_n^{\text{odd}}(t) \cap E_n^{\text{even}}(t)$$

the event that both the odd and even periods are not too short, and

$$E'_n(t) = \{I_{2\lfloor nt/8 \rfloor + 1}(t) \leq n\},$$

the event that we have at least $2\lfloor nt/8 \rfloor + 1$ periods before step n.

We note that for each j, when t is small $J_j(t)$ has expectation $2/t$, so when n is large the above events should all occur with probability close to 1. The following lemma, which we prove later in the section, quantifies this more precisely.

Lemma 3.19. *There exists a constant $\delta > 0$ such that for any $t \in [0, 1]$ and $n \in \mathbb{N}$,*

$$\mathbb{P}(E_n(t)^c) + \mathbb{P}(E'_n(t)^c) \leq \exp(-\delta nt).$$

For now we will work on the event $E_n(t)$. Also define, for $k \in \mathbb{N}$,

$$V_k(t) = \bigcap_{j=1}^{k} A_j(t) \quad \text{and } V'_k(t) = \bigcap_{j=1}^{k} A'_j(t).$$

Our next result translates the probability that we want to bound, which is that of $V_k(t)$, into probabilities of events involving $W(t)$ and $B^{(j)}(t)$. The probabilities on the right are squared, reflecting the fact that we have two Brownian motions (one at time 0 and another at time t) that must both stay positive. Apart from the first period, which is important to retain separately, only the even periods are included, since they are the ones on which the two Brownian motions are mirrored.

Proposition 3.20. *For any $k, n \in \mathbb{N}$ with $n \geq 2k$ and any $t \in [0, 1]$,*

$$\mathbb{P}\big(V_k(t) \cap E_n(t)\big) \leq \mathbb{P}\big(A'_1(t) \cap E_n(t)\big) \\ \cdot \prod_{j=1}^{\lfloor k/2 \rfloor} \mathbb{P}\big(B_i^{(2j)}(t) > 0 \quad \forall i \in [0, J_{2j}(t)) \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big)^2.$$

The proof of this result involves carefully separating out as much independence as possible between the different periods and applying a version of the FKG inequality that specifically works for Brownian motions as opposed to discrete processes. Again we postpone the proof to later in the section in order to continue with our overarching proof of Proposition 3.12.

Next observe that since $B^{(j)}(t)$ is simply an independent Brownian motion started from $W_{I_{j-1}(t)}(t)$, it has the same distribution as W itself over the $(j+1)$th period. This inspires our next proposition, which allows us to telescope the product from Proposition 3.20 back into a statement only about W.

Proposition 3.21. *For any $k, n \in \mathbb{N}$ with $n \geq 2k$ and any $t \in [0, 1]$,*

$$\prod_{j=1}^{k} \mathbb{P}\big(B_i^{(2j)}(t) > 0 \quad \forall i \in [0, J_{2j}(t)) \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big) = \frac{\mathbb{P}\big(\bigcap_{j=1}^{k+1} A'_{2j-1}(t) \cap E_n(t)\big)}{\mathbb{P}(A'_1(t) \cap E_n(t))}.$$

Combining Propositions 3.20 and 3.21, and then using elementary bounds, allows us to prove the following.

Proposition 3.22. *Suppose that $t \in [0, 1]$ and $n \in \mathbb{N}$. Then for any $\lfloor k/2 \rfloor \geq \lfloor nt/8 \rfloor$,* *we have*

$$\mathbb{P}\big(V_k(t) \cap E_n(t)\big) \lesssim \frac{1}{nt^{1/2}}.$$

Leaving the proof of Proposition 3.22 until later, we now observe that

$$\begin{aligned}
\mathbb{P}\big(P_n(0) \cap P_n(t)\big) &= \mathbb{P}\big(P_n(0) \cap P_n(t) \cap E_n(t) \cap E_n'(t)\big) \\
&\qquad + \mathbb{P}\big(P_n(0) \cap P_n(t) \cap (E_n(t)^c \cup E_n'(t)^c)\big) \\
&\le \mathbb{P}\big(V_{2\lfloor nt/8 \rfloor + 1}(t) \cap E_n(t)\big) + \mathbb{P}\big(P_n(0) \cap (E_n(t)^c \cup E_n'(t)^c)\big) \\
&= \mathbb{P}\big(V_{2\lfloor nt/8 \rfloor + 1}(t) \cap E_n(t)\big) + \mathbb{P}\big(P_n(0)\big)\mathbb{P}\big(E_n(t)^c \cup E_n'(t)^c\big)
\end{aligned}$$

where the last equality used the independence of $\mathcal{B}(0)$ and the lengths of the periods at time t. By Proposition 3.22, the first term on the last line above is at most a constant times $1/(nt^{1/2})$, and by Corollary 3.11 and Lemma 3.19, the second term is at most a constant times $n^{-1/2}\exp(-\delta nt)$ for some constant $\delta > 0$. Thus

$$\mathbb{P}\big(P_n(0) \cap P_n(t)\big) \lesssim \frac{1}{nt^{1/2}} + \frac{1}{n^{1/2}}\exp(-\delta nt)$$

and so

$$\int_0^1 \frac{\mathbb{P}(P_n(0) \cap P_n(t))}{t^\gamma}\,\mathrm{d}t \lesssim \frac{1}{n}\int_0^1 t^{-1/2-\gamma}\,\mathrm{d}t + \frac{1}{n^{1/2}}\int_0^1 t^{-\gamma}e^{-\delta nt}\,\mathrm{d}t.$$

For $\gamma < 1/2$, the first integral on the right-hand side above is finite and the second integral (which can be approximated by integrating separately over $(0, 1/n]$ and $(1/n, 1)$) is of order $n^{\gamma-1}$. Therefore, for $\gamma < 1/2$,

$$\int_0^1 \frac{\mathbb{P}(P_n(0) \cap P_n(t))}{t^\gamma}\,\mathrm{d}t \lesssim n^{-1} + n^{\gamma - 3/2} \asymp n^{-1}.$$

Recalling from the start of the section that

$$\mathbb{E}[\Phi_n^\alpha(\gamma)] \le \frac{2}{\mathbb{P}(P_n^\alpha)^2}\int_0^1 \frac{\mathbb{P}(P_n(0) \cap P_n(t))}{t^\gamma}\,\mathrm{d}t,$$

and from Lemma 3.18 that for any $\alpha < 1/2$,

$$\mathbb{P}(P_n^\alpha) \asymp \frac{1}{\sqrt{n}},$$

we have for $\alpha, \gamma < 1/2$ that

$$\mathbb{E}[\Phi_n^\alpha(\gamma)] \lesssim 1.$$

This completes the proof of Proposition 3.12, subject to proving all of the intermediary results above.

Before we begin to prove these results, we will need another elementary lemma as an ingredient in the proof of Proposition 3.20.

Lemma 3.23. *If $(B_s,\, s \geq 0)$ is a Brownian motion, then for any positive $x, y, k \in \mathbb{R}$,*

$$\mathbb{P}_x(B_s \in (0, 2y) \ \ \forall s \leq k) \leq \mathbb{P}_y(B_s \in (0, 2y) \ \ \forall s \leq k).$$

where $\mathbb{P}_x$ is the measure for a Brownian motion started from x.

This is still easy to prove but we do not use induction as in the random walk case. We now proceed with the proofs of Propositions 3.20 and 3.21.

The proof of Proposition 3.20 is practically identical to the random walk case (with Brownian motions replacing random walks of course), but we must use the FKG inequality that applies to Brownian motions.

Proof of Proposition 3.20. Our first step is to move from $A_j(t)$ to $A'_j(t)$. To do so, we go via a third collection of events which we call $\tilde{A}_j(t)$. When j is odd, let $\tilde{A}_j(t) = A'_j(t)$. We have already mentioned that if j is odd, then

$$A_j(t) \subset A'_j(t) = \tilde{A}_j(t).$$

When j is even, define the event

$$\tilde{A}_j(t) = \{\mathcal{B}_s(0) \in (0, 2W_{I_{j-1}(t)}(t)) \ \ \forall s \in [I_{j-1}(t), I_j(t))\}.$$

We claim that when j is even, we also have $A_j(t) \subset \tilde{A}_j(t)$. Indeed, suppose that j is even. We show that if $\omega \notin \tilde{A}_j(t)$ then $\omega \notin A_j(t)$. If $\omega \notin \tilde{A}_j(t)$ then there exists $s \in [I_{j-1}(t), I_j(t))$ such that either $\mathcal{B}_s(0) \leq 0$, in which case clearly $\omega \notin A_j(t)$, or

$$\mathcal{B}_s(0) \geq 2W_{I_{j-1}(t)}(t) = \mathcal{B}_{I_{j-1}(t)}(0) + \mathcal{B}_{I_{j-1}(t)}(t).$$

Then

$$\mathcal{B}_s(0) - \mathcal{B}_{I_{j-1}(t)}(0) \geq \mathcal{B}_{I_{j-1}(t)}(t),$$

so since the increments of $\mathcal{B}_s(t)$ are the negative of the increments of $\mathcal{B}(0)$ during even periods,

$$\mathcal{B}_s(t) - \mathcal{B}_{I_{j-1}(t)}(t) \leq -\mathcal{B}_{I_{j-1}(t)}(t)$$

and therefore $\mathcal{B}_s(t) \leq 0$. Thus $\omega \notin A_j(t)$, establishing our claim. We deduce that, for any $k \in \mathbb{N}$,

$$A_1(t) \cap A_2(t) \cap \ldots \cap A_k(t) \subset \tilde{A}_1(t) \cap \tilde{A}_2(t) \cap \ldots \cap \tilde{A}_k(t). \tag{3.7}$$

Note that the increments of $\mathcal{B}_s(0)$ on even periods are independent of the whole process $W_s(t)$. Combining this fact with Lemma 3.23, we have

$$\mathbb{P}\big(\tilde{A}_1(t) \cap \tilde{A}_2(t) \cap \ldots \cap \tilde{A}_k(t) \big| \mathcal{F}_{I(t)}\big) \leq \mathbb{P}\big(A'_1(t) \cap A'_2(t) \cap \ldots \cap A'_k(t) \big| \mathcal{F}_{I(t)}\big) \tag{3.8}$$

for any $k \in \mathbb{N}$, where $\mathcal{F}_{I(t)} = \sigma(I_j(t), j \geq 0)$. Combining (3.7) and (3.8) and taking expectations to remove the conditioning, for any $k \in \mathbb{N}$ we have

$$\mathbb{P}(V_k(t) \cap E_n(t)) \leq \mathbb{P}(V'_k(t) \cap E_n(t)).$$

Repeated use of the definition of conditional probability, followed by ignoring the odd terms for $j \geq 3$, gives

$$\begin{aligned}\mathbb{P}\big(V_k(t) \cap E_n(t)\big) &\leq \mathbb{P}\big(A'_1(t) \cap E_n(t)\big) \cdot \prod_{j=2}^{k} \mathbb{P}\big(A'_j(t) \,\big|\, V'_{j-1}(t) \cap E_n(t)\big) \\ &\leq \mathbb{P}\big(A'_1(t) \cap E_n(t)\big) \cdot \prod_{j=1}^{\lfloor k/2 \rfloor} \mathbb{P}\big(A'_{2j}(t) \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big). \end{aligned} \tag{3.9}$$

Recalling that $B^{(2j)}$ starts from $W_{I_{2j-1}(t)}$ and that

$$\begin{aligned} A'_{2j}(t) &= \{B^{(2j)}_s(t) \in (0, 2W_{I_{2j-1}(t)}(t)) \;\; \forall s \in [0, J_{2j}(t))\} \\ &= \{B^{(2j)}_s(t) > 0 \;\; \forall s \in [0, J_{2j}(t))\} \\ &\qquad\qquad \cap \{B^{(2j)}_s(t) < 2W_{I_{2j-1}(t)}(t) \;\; \forall s \in [0, J_{2j}(t))\}, \end{aligned}$$

we observe that $A'_{2j}(t)$ is the intersection of two events, one that is increasing and the other decreasing. Thus we can apply a version of the Brownian FKG inequality, Corollary 3.5, to obtain

$$\begin{aligned} &\mathbb{P}\big(A'_{2j}(t) \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big) \\ &\qquad \leq \mathbb{P}\big(B^{(2j)}_s(t) > 0 \;\; \forall s \in [0, J_{2j}(t)) \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big) \\ &\qquad\qquad \cdot \mathbb{P}\big(B^{(2j)}_s(t) < 2W_{I_{2j-1}(t)}(t) \;\; \forall s \in [0, J_{2j}(t)) \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big) \\ &\qquad = \mathbb{P}\big(B^{(2j)}_s(t) > 0 \;\; \forall s \in [0, J_{2j}(t)) \,\big|\, V'_{2j-1}(t) \cap E_n(t)\big)^2, \end{aligned}$$

where the equality follows from symmetry about $W_{I_{2j-1}(t)}(t)$ (recalling that $B_0^{(2j)}(t) = W_{I_{2j-1}(t)}(t)$). Substituting this into (3.9), we have shown that

$$\mathbb{P}\big(V_k(t) \cap E_n(t)\big) \leq \mathbb{P}\big(A_1'(t) \cap E_n(t)\big) \cdot \prod_{j=1}^{\lfloor k/2 \rfloor} \mathbb{P}\big(B_i^{(2j)}(t) > 0 \ \ \forall i \in [0, J_{2j}(t)) \,\big|\, V_{2j-1}'(t) \cap E_n(t)\big)^2$$

as required. $\square$

The proof of Proposition 3.21 below is an induction argument that relies on the key idea that each of the $B_s^{(2j)}(t)$ (over j) are independent of $W(t)$ but have the same distribution. Thus we can just think about the next odd period of $W(t)$ rather than $B_s(t)$. See Figure 3-4.

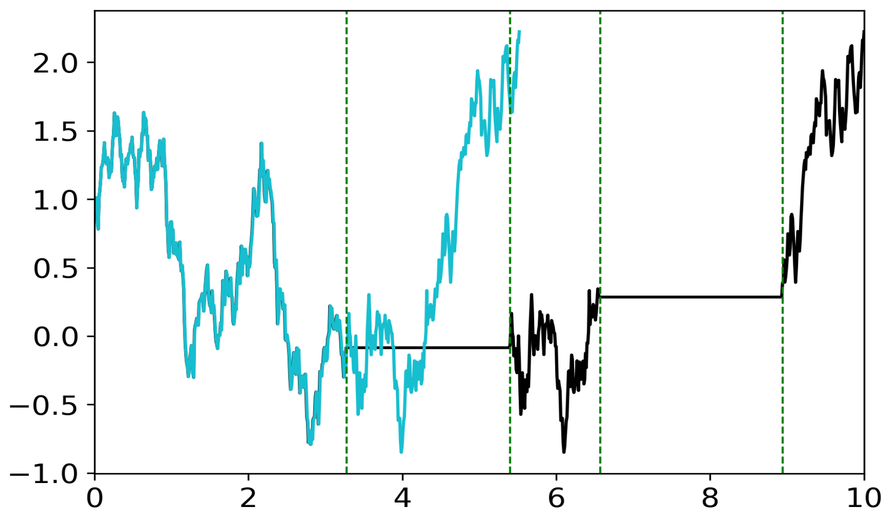

Figure 3-4: A realisation of $W(t)$ (black) and its contracted version (cyan) up until Brownian time 10. The green marks the position of the reflections that have occured up until dynamical time t.

Proof of Proposition 3.21. We work by induction on k. For $k = 1$, we have

$$\mathbb{P}\big(B_s^{(2)}(t) > 0 \ \ \forall s \in [0, J_2(t)) \,\big|\, V_1'(t) \cap E_n(t)\big) = \frac{\mathbb{P}\big(\{B_s^{(2)}(t) > 0 \ \ \forall s \in [0, J_2(t))\} \cap A_1'(t) \cap E_n(t)\big)}{\mathbb{P}\big(A_1'(t) \cap E_n(t)\big)}.$$

On the event $A_1'(t) \cap E_n(t)$, the distribution of $(B_s^{(2)}(t))_{s \in [0, J_2(t))}$ is identical to that of

$(W_{I_2(t)+s}(t))_{s\in[0,J_3(t))}$, and therefore

$$\mathbb{P}\big(B_s^{(2)}(t) > 0 \ \ \forall s \in [0, J_2(t)) \,\big|\, V_1'(t) \cap E_n(t)\big) = \frac{\mathbb{P}\big(A_3'(t) \cap A_1'(t) \cap E_n(t)\big)}{\mathbb{P}\big(A_1'(t) \cap E_n(t)\big)},$$

establishing the claim in the case $k = 1$. The general case is very similar: assuming that the claim holds for $k-1$, we have

$$\begin{aligned}&\prod_{j=1}^{k} \mathbb{P}\big(B_s^{(2j)}(t) > 0 \ \ \forall s \in [0, J_{2j}(t)) \,\big|\, V_{2j-1}'(t) \cap E_n(t)\big) \\ &\quad = \frac{\mathbb{P}\big(\bigcap_{j=1}^{k} A_{2j-1}'(t) \cap E_n(t)\big)}{\mathbb{P}\big(A_1'(t) \cap E_n(t)\big)} \mathbb{P}\big(B_0^{(2k)}(t) > 0 \ \ \forall s \in [0, J_{2k}(t)) \,\big|\, V_{2k-1}'(t) \cap E_n(t)\big).\end{aligned}$$

Considering the last term on the right-hand side above, we note that $B^{(2k)}(t)$ is independent of $A_{2j}'(t)$ given $A_{2j-1}'(t)$ for all $j < k$, and therefore the above equals

$$\begin{aligned}&\frac{\mathbb{P}\big(\bigcap_{j=1}^{k} A_{2j-1}'(t) \cap E_n(t)\big)}{\mathbb{P}\big(A_1'(t) \cap E_n(t)\big)} \mathbb{P}\left(B_s^{(2k)}(t) > 0 \ \ \forall s \in [0, J_{2k}(t)) \,\middle|\, \bigcap_{j=1}^{k} A_{2j-1}'(t) \cap E_n(t)\right) \\ &\qquad = \frac{\mathbb{P}\big(\{B_s^{(2k)}(t) > 0 \ \ \forall s \in [0, J_{2k}(t))\} \cap \bigcap_{j=1}^{k} A_{2j-1}'(t) \cap E_n(t)\big)}{\mathbb{P}\big(A_1'(t) \cap E_n(t)\big)}.\end{aligned}$$

Provided that $2k \le n$, on the event $\bigcap_{j=1}^{k} A_{2j-1}'(t) \cap E_n(t)$, the law of $(B_s^{(2k)}(t))_{s\in[0,J_{2k}(t))}$ is identical to that of $(W_{I_{2k}(t)+s}(t))_{s\in[0,J_{2k+1}(t))}$, and therefore

$$\mathbb{P}\left(\Big\{B_s^{(2k)}(t) > 0 \ \ \forall s \in [0, J_{2k}(t))\Big\} \cap \bigcap_{j=1}^{k} A_{2j-1}'(t) \cap E_n(t)\right) = \mathbb{P}\left(\bigcap_{j=1}^{k+1} A_{2j-1}'(t) \cap E_n(t)\right)$$

which establishes the claim for k, completing the proof. $\square$

The proof of our third proposition in this Section, Proposition 3.22, does not contain any major ideas; it simply combines the results above with some elementary approximations.

Proof of Proposition 3.22. Combining Propositions 3.20 and 3.21, we have

$$\mathbb{P}\big(V_k(t) \cap E_n(t)\big) \le \frac{\mathbb{P}\big(\bigcap_{j=1}^{\lfloor k/2 \rfloor + 1} A_{2j-1}'(t) \cap E_n(t)\big)^2}{\mathbb{P}(A_1'(t) \cap E_n(t))}.$$

Recalling that $A_{2j-1}'(t)$ requires that $W_s(t)$ is positive on the $(2j-1)$th period, whereas

$W_s(t)$ is constant on even periods, we note that

$$\bigcap_{j=1}^{\lfloor k/2\rfloor+1} A'_{2j-1}(t) = \{W_s(t) > 0 \ \ \forall s < I_{2\lfloor k/2\rfloor+1}(t)1\}$$

and therefore

$$\mathbb{P}\big(V_k(t)\cap E_n(t)\big) \le \frac{\mathbb{P}\big(\{W_s(t)>0 \ \ \forall s < I_{2\lfloor k/2\rfloor+1}(t)\}\cap E_n(t)\big)^2}{\mathbb{P}(A'_1(t)\cap E_n(t))}.$$

Now, $W_s(t)$ is simply a Brownian motion during odd periods, and constant on even periods. Thus the probability that it stays positive up to Brownian-time $I_{2\lfloor k/2\rfloor+1}(t)$ is exactly the probability that a Brownian motion stays positive up to Brownian-time $J_1(t)+J_3(t)+\ldots+J_{2\lfloor k/2\rfloor+1}(t)$. We deduce that

$$\begin{aligned}\mathbb{P}\big(V_k(t)\cap E_n(t)\big) &\le \frac{\mathbb{P}\big(\{\mathcal{B}_s(t)>0 \ \ \forall s < J_1(t)+J_3(t)+\ldots+J_{2\lfloor k/2\rfloor+1}(t)\}\cap E_n(t)\big)^2}{\mathbb{P}(A'_1(t)\cap E_n(t))}\\ &\le \frac{\mathbb{P}\big(\mathcal{B}_s(t)>0 \ \ \forall s \le J_1(t)+J_3(t)+\ldots+J_{2\lfloor k/2\rfloor+1}(t) \,\big|\, E_n(t)\big)^2}{\mathbb{P}\big(A'_1(t)\,\big|\,E_n(t)\big)}.\end{aligned}$$

On the event $E_n(t)\subset E_n^{\mathrm{odd}}(t)$, we have

$$J_1(t)+J_3(t)+\ldots+J_{2\lfloor nt/8\rfloor+1}(t) \ge J_3(t)+J_5(t)+\ldots+J_{2\lfloor nt/8\rfloor+1}(t) \ge n/8,$$

and therefore for any $k\ge nt/4$,

$$\mathbb{P}\big(V_k(t)\cap E_n(t)\big) \le \frac{\mathbb{P}\big(\mathcal{B}_s(t)>0 \ \ \forall s < n/8\big)^2}{\mathbb{P}\big(A'_1(t)\,\big|\,E_n(t)\big)} = \frac{\mathbb{P}\big(\mathcal{B}_s(0)>0 \ \ \forall s < n/8\big)^2}{\mathbb{P}\big(A'_1(t)\big)}, \qquad (3.10)$$

where the equality holds by stationarity of $\mathcal{B}(t)$ and the independence of $A'_1(t)$ and $E_n(t)$ (since $E_n(t)$ only involves periods 2 and later). We know from Corollary 3.11 that

$$\mathbb{P}\big(\mathcal{B}_s(0)>0 \ \ \forall s<n/8\big) \asymp n^{-1/2},$$

and we claim that

$$\mathbb{P}(A'_1(t)) \gtrsim t^{1/2}.$$

To see this, note that we are asking for the probability that a Brownian motion starting from one stays positive for an $Exp(t/2)$ amount of Brownian time. Denoting this length of time by T, we have

$$\mathbb{P}(A_1'(t)) \geq \mathbb{P}(\mathcal{B}_s > 0 \;\forall\; 0 \leq s \leq t^{-1})\mathbb{P}(T < t^{-1}) \asymp \sqrt{t}$$

where we've used Corollary 3.11 to bound the first term and the second term is a non-zero constant since (using a $y = tx$ substitution)

$$\mathbb{P}(T < t^{-1}) = \int_0^{t^{-1}} \frac{t}{2} e^{-\frac{tx}{2}} \, \mathrm{d}x = \frac{1}{2} \int_0^1 e^{-\frac{y}{2}} \, \mathrm{d}y > 0.$$

Substituting our approximations into (3.10), we have shown that for any $k \geq nt/4$,

$$\mathbb{P}\big(V_k(t) \cap E_n(t)\big) \lesssim \frac{1}{nt^{1/2}}$$

as required. □

We now proceed with the proofs of our minor lemmas.

Proof of Lemma 3.19. We begin by considering $E_n^{\text{odd}}(t)$. In order for $E_n^{\text{odd}}(t)^c$ to occur, the sum of $\lfloor nt/8 \rfloor$ independent exponential random variables of rate $t/2$ must be smaller than $n/8$. It is equivalent to think about the probability that we have at least $\lfloor nt/8 \rfloor$ Poisson points in the interval $[0, n/8]$. The number of points in said interval are $X \sim Po((t/2)(n/8)) = Po(nt/16))$ distributed. We can use the moment generating function of X to deduce

$$\mathbb{E}[e^{(\log 2)X}] = e^{(nt/16)(e^{\log 2} - 1)} = e^{nt/16}$$

and thus by Markov's inequality

$$\mathbb{P}(X \geq \lfloor nt/8 \rfloor) \leq e^{nt/16} e^{-(\log 2)\lfloor nt/8 \rfloor} \leq e^{(nt/16) - (\log 2)(nt/8 - 1)} = 2e^{-(2\log 2 - 1)nt/16}.$$

This proves the required decay for $\mathbb{P}(E_n^{\text{odd}}(t)^c)$, and $\mathbb{P}(E_n^{\text{even}}(t)) = \mathbb{P}(E_n^{\text{odd}}(t))$. Noting that $I_j(t)$ is a sum of j independent Exponential random variables of rate $t/2$, we have $\mathbb{P}(E_n'(t)^c) = \mathbb{P}(Y < 2\lfloor nt/8 \rfloor + 1)$ where $Y \sim Po(nt/2)$. Using a Chernoff bound argument again with the moment generating function (with $-\log 2$ rather than $\log 2$)

it can be seen that

$$\begin{aligned}\mathbb{P}(Y < 2\lfloor nt/8\rfloor + 1) &\leq \mathbb{P}(e^{-(\log 2)Y} \geq e^{-(\log 2)(2\lfloor nt/8\rfloor+1)})\\ &\leq \mathbb{E}[e^{-(\log 2)Y}]e^{(\log 2)(2\lfloor nt/8\rfloor+1)}\\ &\leq e^{-(nt/4)}e^{(\log 2)(nt/4)+\log 2}\\ &\leq 2e^{-(1-\log 2)nt/4}\end{aligned}$$

concluding the proof. □

Proof of Lemma 3.23. By symmetry, we can W.L.O.G consider $x < y$. Let $(B_s)_s$ be a Brownian motion started from zero. Define $Y_s = y + B_s$, and

$$T = \inf\{s \ : \ Y_s = (x+y)/2\}.$$

Let $X_s = x - B_s$ for $s \leq T$, with $X_s = Y_s$ thereafter. Then (X_s) and (Y_s) are coupled Brownian motions starting from x and y respectively, and (Y_s) cannot exit the interval $(0, 2y)$ before (X_s) does. Therefore if (X_s) stays in this interval up to Brownian time k, then so does (Y_s), implying the result. □

3.6 Proof of Proposition 3.17: influences of P_n

In this section we give estimates on the influences of each Brownian time m of P_n. We claim that given any fixed $\varepsilon, x > 0$ and real $0 \leq m < n - \varepsilon$ that

$$\mathcal{I}_m^x(P_n) \asymp \frac{n-m}{n^{3/2}},$$

where $\mathcal{I}_m^x(P_n)$ is the probability that reflecting a Brownian motion after Brownian time m will change whether or not P_n occurs. We will keep n fixed and say "m is pivotal" as shorthand for "m is pivotal for P_n". Note that we have excluded $m \geq n$ here as it is trivial to see that the influence is zero in these cases. We also have forced $m \leq n - \varepsilon$ because the case where $n - m \to 0$ needs to be treated differently, but it is unneeded for our work.

3.6.1 Translating $\mathcal{I}_m^x(P_n)$ into properties of the Brownian Motion

To reduce the amount of work we will take advantage of the fact that

$$\mathcal{I}_m^x(P_n) = \mathbb{P}_x(m \text{ is pivotal}) = 2\mathbb{P}_x(\{m \text{ is pivotal}\} \cap P_n), \tag{3.11}$$

which holds since the reflected Brownian motion has the same law as a Brownian motion. To be precise let $(\bar{B}_s)$ be the Brownian motion reflected at Brownian time m, that is, $\bar{B}_s = B_s$ for $s \leq m$ and

$$\bar{B}_s = 2B_m - B_s$$

for $s > m$. Then, noting that $\bar{\bar{B}}_s = B_s$ we get

$$\begin{aligned}
&\mathbb{P}_x(\{m \text{ is pivotal}\} \cap P_n) \\
&= \mathbb{P}_x(\{B_s > 0 \ \forall s \leq n\} \cap \{\exists s \leq n \ \bar{B}_s = 0\}) \\
&= \mathbb{P}_x(\{\bar{B}_s > 0 \ \forall s \leq n\} \cap \{\exists s \leq n \ \bar{\bar{B}}_s = 0\}) \\
&= \mathbb{P}_x(\{\bar{B}_s > 0 \ \forall s \leq n\} \cap \{\exists s \leq n \ B_s = 0\}) \\
&= \mathbb{P}_x(\{m \text{ is pivotal}\} \cap P_n^c).
\end{aligned}$$

implying (3.11).

We now write down an explicit condition for the event $\{m \text{ is pivotal}\} \cap P_n$ to occur. We claim that for $0 \leq m < n$,

$$\{m \text{ is pivotal}\} \cap P_n = \{B_s > 0 \ \ \forall 0 \leq s \leq n\} \cap \Big\{ \sup_{m \leq s \leq n} B_s \geq 2B_m \Big\}. \tag{3.12}$$

In words, m is pivotal and P_n holds if and only if B stays positive for the first n Brownian time, and hits $2B_m$ between Brownian times m and n.

To see why this is true, call the path of B up until Brownian time m the *first portion* of the process, and the path from Brownian time m until n the *second portion*. Of course P_n entails that both portions remain positive. In order for m to be pivotal, we also need that when we reflect the second portion of the path about B_m, the second portion no longer remains positive. This holds if and only if the second portion (prior to reflecting) hits $2B_m$. See Figure 3-5.

We now split the event that m is pivotal over the possible values of B_m, via an integral over the Brownian density. Writing $\mathbb{P}_z$ for the probability measure under which our process starts from z, by (3.11) and (3.12)

$$\begin{aligned}
\mathcal{I}_m^x(P_n) = 2 \int_0^\infty & \mathbb{P}_x\Big(\inf_{s \leq m} B_s > 0, \, B_m \in \mathrm{d}z \Big) \\
& \cdot \mathbb{P}_z\Big(\Big\{ \inf_{s \leq n-m} B_s > 0 \Big\} \cap \Big\{ \sup_{s \leq n-m} B_s \geq 2z \Big\} \Big).
\end{aligned} \tag{3.13}$$

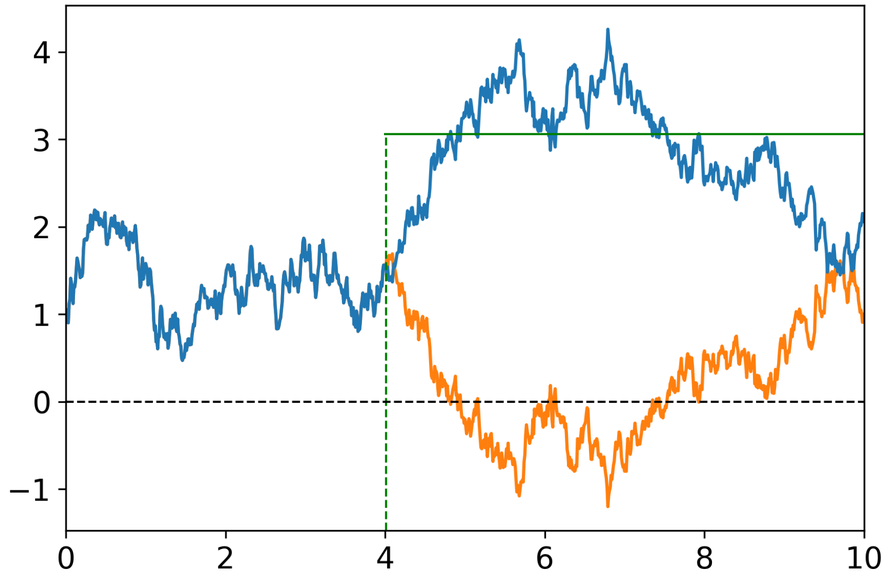

Figure 3-5: A realisation of B where reflecting at $m = 4$ is pivotal for P_{10}. The solid green line is at height $2B_4$. Note that B must be positive on $[0, 4]$.

Note that we do not use the Ballot theorem as we did in the random walk case.

3.6.2 A lower bound on the influences of P_n

Define the events

$$L = L(m,n) = \Big\{ \inf_{s \le n-m} B_s > 0 \Big\} \quad \text{and} \quad U = U(m,n,z) = \Big\{ \sup_{s \le n-m} B_s \ge 2z \Big\}. \quad (3.14)$$

Let

$$l(m,n) = \left\lfloor \frac{\sqrt{n-m}}{2} \right\rfloor \wedge \left\lfloor \frac{\sqrt{m}}{2} \right\rfloor.$$

We want to bound $\mathbb{P}_z(L \cap U)$ from below when $z \le l(m,n)$. We now use Corollary 3.11 to prove a further corollary.

Corollary 3.24. *If $0 \le z \le \sqrt{n-m}$ and $0 \le m \le n - \varepsilon$ then as n gets large*

$$\mathbb{P}_z(L(m,n)) \asymp \frac{z}{\sqrt{n-m}}$$

and if $0 \le z \le l(m,n)$ and $0 \le m \le n - \varepsilon$ then as n gets large

$$\mathbb{P}_z(U(m,n,z)) \asymp 1.$$

Proof. Noting that $z^2 \leq n-m$, Corollary 3.11 instantly gives us

$$\mathbb{P}_z(L(m,n)) = \mathbb{P}_0(\inf_{s\leq n-m} B_s > -z) \asymp \frac{z}{\sqrt{n-m}}.$$

The second part follows the same idea as the proof of Corollary 3.11, as well as utilising Brownian scaling and $z \leq \sqrt{n-m}$, to obtain:

$$\mathbb{P}_z(U) = \mathbb{P}_0(|B_1| > \frac{2z}{\sqrt{n-m}}) = 2\int_{2z(n-m)^{-1/2}}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x \geq 2\int_2^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x > 0$$

and clearly $\mathbb{P}_z(U) \leq 1$ so the proof is complete. $\square$

Lemma 3.25. *For $z \in [0, l(m,n)]$ with $0 \leq m \leq n-\varepsilon$, we have as n gets large that*

$$\mathbb{P}_z\Big(L(m,n) \cap U(m,n,z)\Big) \gtrsim \frac{z}{\sqrt{n-m}}.$$

Proof. L and U are both increasing events with respect to Brownian paths, therefore we can use the Brownian FKG inequality, Proposition 3.4, to obtain

$$\mathbb{P}_z(L \cap U) \geq \mathbb{P}_z(L)\mathbb{P}_z(U).$$

The result now follows from Corollary 3.24. $\square$

Now we have all the information required to generate our lower bound for Proposition 3.17. We split our work into the cases $m \geq 2$ and $m < 2$ for technical reasons, remembering that n is large.

Starting with $m \geq 2$, we use Lemmas 3.8, 3.25 and Corollary 3.7 with equation (3.13)

$$\begin{aligned}
\mathcal{I}_m^x(P_n) &= 2\int_0^{\infty} \mathbb{P}_x\Big(\inf_{s\leq m} B_s > 0,\, B_m \in \mathrm{d}z\Big) \\
&\qquad\qquad \cdot \mathbb{P}_z\Big(\big\{\inf_{s\leq n-m} B_s > 0\big\} \cap \big\{\sup_{s\leq n-m} B_s \geq 2z\big\}\Big) \\
&\gtrsim \int_0^{l(m,n)} \mathbb{P}_x\Big(\inf_{s\leq m} B_s > 0,\, B_m \in \mathrm{d}z\Big) \\
&\qquad\qquad \cdot \mathbb{P}_z\Big(\big\{\inf_{s\leq n-m} B_s > 0\big\} \cap \big\{\sup_{s\leq n-m} B_s \geq 2z\big\}\Big) \\
&\gtrsim \int_0^{l(m,n)} \frac{2}{\sqrt{2\pi m}} \exp\left(-\frac{z^2+x^2}{2m}\right) \sinh\left(\frac{zx}{m}\right) \frac{z}{\sqrt{n-m}}\,\mathrm{d}z \\
&\gtrsim \int_0^{l(m,n)} \frac{z^2 x}{m^{3/2}(n-m)^{1/2}} \exp\left(-\frac{z^2+x^2}{2m}\right) \mathrm{d}z.
\end{aligned}$$

If $z \le l(m,n)$ and $m \ge 2$ then

$$\exp\left(-\frac{z^2+x^2}{2m}\right) \ge \exp\left(-\frac{(m/4)+x^2}{2m}\right) = \exp\left(-\frac{1+2x^2}{8}\right) \ge C_x > 0$$

so the exponential term is at least a constant that depends on x. So our lower bound for the pivotal probability is

$$\asymp \int_0^{l(m,n)} \frac{z^2 x}{m^{3/2}(n-m)^{1/2}}\,\mathrm{d}z \asymp \frac{x l(m,n)^3}{m^{3/2}(n-m)^{1/2}}.$$

If $m \le n/2$, then the right-hand side above is of order $n^{-1/2}$, and if $m > n/2$, it is of order $(n-m)/n^{3/2}$ as required for this case. This is true since $n - m \ge \varepsilon > 0$.

For the case of $m < 2$ we need to be more careful as our lower bounds do not behave nicely for small m. We have

$$\begin{aligned}
\mathcal{I}_m^x(P_n) &= 2\int_0^\infty \mathbb{P}_x\Big(\inf_{s\le m} B_s > 0,\, B_m \in \mathrm{d}z\Big) \\
&\qquad\qquad\qquad \cdot \mathbb{P}_z\Big(\big\{\inf_{s\le n-m} B_s > 0\big\} \cap \big\{\sup_{s\le n-m} B_s \ge 2z\big\}\Big) \\
&\gtrsim \int_{x/2}^{2x} \mathbb{P}_x\Big(\inf_{s\le m} B_s > 0,\, B_m \in \mathrm{d}z\Big) \cdot \mathbb{P}_z\Big(\big\{\inf_{s\le n} B_s > 0\big\} \cap \big\{\sup_{s\le n-2} B_s \ge 2z\big\}\Big) \\
&\ge \mathbb{P}_{x/2}\Big(\big\{\inf_{s\le n} B_s > 0\big\} \cap \big\{\sup_{s\le n-2} B_s \ge 4x\big\}\Big) \int_{x/2}^{2x} \mathbb{P}_x\Big(\inf_{s\le m} B_s > 0,\, B_m \in \mathrm{d}z\Big) \\
&= \mathbb{P}_{x/2}\Big(\big\{\inf_{s\le n} B_s > 0\big\} \cap \big\{\sup_{s\le n-2} B_s \ge 4x\big\}\Big) \\
&\qquad\qquad\qquad \cdot \mathbb{P}_x\Big(\inf_{s\le m} B_s > 0,\, B_m \in (x/2, 2x)\Big).
\end{aligned}$$

We now work on each of the probabilities separately. Brownian scaling gives

$$\begin{aligned}
&\mathbb{P}_x\Big(\inf_{s\le m} B_s > 0, B_m \in (x/2, 2x)\Big) \\
&\qquad = \mathbb{P}_{x/\sqrt{m}}\left(\inf_{s\le 1} B_s > 0,\, B_1 \in \Big(\frac{x}{2\sqrt{m}}, \frac{2x}{\sqrt{m}}\Big)\right) \\
&\qquad = \mathbb{P}_{x/\sqrt{m}}\left(\inf_{s\le 1} B_s > 0,\, B_1 \in \Big(\frac{x}{\sqrt{m}} - \frac{x}{2\sqrt{m}}, \frac{x}{\sqrt{m}} + \frac{x}{\sqrt{m}}\Big)\right) \\
&\qquad \ge \mathbb{P}_{x/\sqrt{m}}\left(\inf_{s\le 1} B_s > 0,\, B_1 \in \Big(\frac{x}{\sqrt{m}} - \frac{x}{3}, \frac{x}{\sqrt{m}} + \frac{x}{3}\Big)\right) \\
&\qquad \ge \mathbb{P}_{x/\sqrt{2}}\left(\inf_{s\le 1} B_s > 0,\, B_1 \in \Big(\frac{x}{\sqrt{2}} - \frac{x}{3}, \frac{x}{\sqrt{2}} + \frac{x}{3}\Big)\right) \\
&\qquad \ge C > 0
\end{aligned}$$

where we've used that $m < 2$ and have focused on making the region that B_1 can lie in small but constant in radius. This last probability is obviously a non-zero constant (as x is constant). We now proceed to the other probability

$$\mathbb{P}_{x/2}\Big(\big\{\inf_{s\le n} B_s > 0\big\} \cap \big\{\sup_{s\le n-2} B_s \ge 4x\big\}\Big)$$

which is equal to

$$\mathbb{P}_{x/2}\big(\inf_{s\le n} B_s > 0\big) - \mathbb{P}_{x/2}\big(B_s \in (0,4x)\ \forall s \le n-2,\ B_s > 0\ \forall s \in (n-2,n]\big)$$

which is at least

$$\mathbb{P}_{x/2}\big(\inf_{s\le n} B_s > 0\big) - \mathbb{P}_{x/2}\big(B_s \in (0,4x)\ \forall s \le n-2\big).$$

The first probability is of order $n^{-1/2}$ by Corollary 3.11, while using a result just after [35, Theorem 7.43] shows that the second probability is

$$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\exp\left(-\frac{(2k+1)^2\pi^2}{32x^2}(n-2)\right)\sin\left(\frac{(2k+1)\pi}{8}\right)$$

which is of order $\exp(-\pi^2 n/(32x^2))$ (as the $k = 0$ term dominates), so the above difference is still of order $n^{-1/2}$. Combining these probabilities together gives that for $m < 2$:

$$\mathcal{I}_m^x(P_n) \gtrsim C\left(\frac{1}{\sqrt{n}} - e^{-cn}\right) \asymp n^{-\frac{1}{2}} \asymp \frac{n-m}{n^{3/2}}$$

since $m < 2$, as required.

3.6.3 An upper bound on the influences of P_n

Using Lemma 3.18, we immediately see that for $m < n/2$ and fixed $x > 0$

$$\mathcal{I}_m^x(P_n) = 2\mathbb{P}_x(\{m \text{ is pivotal}\} \cap P_n) \le 2\mathbb{P}_x(P_n) \asymp n^{-1/2} \asymp (n-m)n^{-3/2}$$

so we now only need to focus on the $m \ge n/2$ case. We shall need the following Lemma:

Lemma 3.26. *There exists $K > 0$ such that for all $c > 0$ and $r \ge 0$*

$$\int_0^\infty z^r \exp\Big(-\frac{z^2}{c}\Big)\,\mathrm{d}z \le Kc^{(r+1)/2}.$$

Proof. A substitution of $y = z/\sqrt{c}$ does the trick:

$$\int_0^\infty z^r \exp\Big(-\frac{z^2}{c}\Big)\,\mathrm{d}z = c^{(r+1)/2}\int_0^\infty y^r \exp\big(-y^2\big)\,\mathrm{d}y \lesssim c^{(r+1)/2}. \qquad \square$$

We will now bound (3.13) from above. This direction is far more involved as we need to consider the entire sum; for the lower bound we could restrict to just the values of z that gave the biggest contribution. We recall the definitions of L and U from (3.14).

Similarly to the random walk case, we get an effective upper bound on (3.13) by splitting the integral depending on the value of z. We again have three regimes; those being $[0, M']$, $[M', m^{3/4}]$ and $[m^{3/4}, \infty]$, where $M' := m^{3/4} \wedge \sqrt{n-m}$ which equals $\sqrt{n-m}$ provided $m > n/2$ and $n \geq 2$ (the former is assumed, and the latter is true as we only care about large n). These three regimes are the same as the ones in the random walk case, just without adjustments due to continuity and Brownian motions are unbounded in finite (Brownian) time, unlike random walks. From (3.13) we have;

$$\begin{aligned}\mathcal{I}^x_m(P_n) =& 2\int_0^{M'} \mathbb{P}_x\Big(\inf_{s\leq m} B_s > 0,\, B_m \in \mathrm{d}z\Big)\mathbb{P}_z(L\cap U)\\ &+ 2\int_{M'}^{m^{3/4}} \mathbb{P}_x\Big(\inf_{s\leq m} B_s > 0,\, B_m \in \mathrm{d}z\Big)\mathbb{P}_z(L\cap U)\\ &+ 2\int_{m^{3/4}}^{\infty} \mathbb{P}_x\Big(\inf_{s\leq m} B_s > 0,\, B_m \in \mathrm{d}z\Big)\mathbb{P}_z(L\cap U). \end{aligned} \tag{3.15}$$

We label the three integrals in (3.15) by (3.15 i), (3.15 ii) and (3.15 iii).

Starting with (3.15 iii), we bound $\mathbb{P}_z(L\cap U)$ above by 1 and bound $\mathbb{P}_x\Big(\inf_{s\leq m} B_s > 0,\, B_m \in \mathrm{d}z\Big)$ above by ignoring the inf condition. As x is fixed, and $m > n/2$, we can pick n such that $m^{3/4} - x \geq m^{3/4}/2$. This gives for such an n:

$$\begin{aligned}(3.15\,\text{iii}) &\leq 2\int_{m^{3/4}}^\infty \mathbb{P}_x(B_m \in \mathrm{d}z) = 2\mathbb{P}_0(B_m > m^{3/4} - x)\\ &\leq 2\mathbb{P}_0(B_m > m^{3/4}/2) = 2\mathbb{P}_0(B_1 > m^{1/4}/2).\end{aligned}$$

By Lemma 3.9 we have for sufficiently large n that

$$\mathbb{P}_0(B_1 > m^{1/4}/2) \leq \exp(-m^{1/2}/8) \leq \exp(-n^{1/2}/8\sqrt{2})$$

as $m > n/2$. In other words,

$$(3.15\,\text{iii}) \lesssim \exp(-n^{1/2}/8\sqrt{2})$$

which decays super-polynomially, is at most a constant times $(n-m)n^{-3/2}$. This is true since $n-m \geq \varepsilon > 0$. Note that the constant in this portion does not depend on x.

We move on to (3.15 i), recalling that $m > n/2 \geq 1$. We bound $\mathbb{P}_z(L \cap U)$ above by $\mathbb{P}_z(L)$ and then apply Corollary 3.24 alongside Corollary 3.7 and Lemma 3.8 to obtain

$$\begin{aligned}
&\int_0^{\sqrt{n-m}} \mathbb{P}_x\Big(\inf_{s\leq m} B_s > 0,\, B_m \in \mathrm{d}z\Big)\mathbb{P}_z(L\cap U) \\
&\leq \int_0^{\sqrt{n-m}} \mathbb{P}_x\Big(\inf_{s\leq m} B_s > 0,\, B_m \in \mathrm{d}z\Big)\mathbb{P}_z(L) \\
&\lesssim \int_0^{\sqrt{n-m}} \frac{zx}{m}\frac{1}{m^{1/2}}\exp\left(\frac{z^2x^2}{6m^2} - \frac{z^2+x^2}{2m}\right)\frac{z}{(n-m)^{1/2}}\,\mathrm{d}z \\
&\leq \frac{x}{m^{3/2}(n-m)^{1/2}}\int_0^{\sqrt{n-m}} z^2\,\mathrm{d}z \\
&\lesssim \frac{n-m}{m^{3/2}} \asymp \frac{n-m}{n^{3/2}}
\end{aligned}$$

which holds as $m > n/2$ (so $m \asymp n$), as required. Note that the implicit constant depends on x via the polynomial x, which is trivially increasing, as stated in the statement of Proposition 3.17. The main non-trivial step is from the third line to the fourth line we require the exponential term to be ≤ 1, in other words we need the exponent to be non-positive. This definitely occurs when $z^2 \leq 3m$ as in this case

$$\frac{z^2x^2}{6m^2} - \frac{z^2+x^2}{2m} \leq \frac{x^2}{2m} - \frac{z^2+x^2}{2m} = -\frac{z^2}{2m} \leq 0$$

and we know $z^2 \leq n-m \leq m$ (based on integral limits and the fact that $m > n/2$).

For (3.15 ii), we bound $\mathbb{P}_z(L \cap U)$ above by $\mathbb{P}_z(U)$ instead of $\mathbb{P}_z(L)$. We get

$$\begin{aligned}
&\int_{\sqrt{n-m}}^{m^{3/4}} \mathbb{P}_x\Big(\inf_{s\leq m} B_s > 0,\, B_m \in \mathrm{d}z\Big)\mathbb{P}_z(L\cap U) \\
&\qquad\qquad \leq \int_{\sqrt{n-m}}^{m^{3/4}} \mathbb{P}_x\Big(\inf_{s\leq m} B_s > 0,\, B_m \in \mathrm{d}z\Big)\mathbb{P}_z(U),
\end{aligned}$$

and by Lemma 3.10, Brownian scaling, symmetry, and then Lemma 3.9 we obtain

$$\begin{aligned}
\mathbb{P}_z(U) = \mathbb{P}_0(\sup_{s\leq n-m} B_s \geq z) = \mathbb{P}_0(|B_1| \geq z(n-m)^{-1/2}) &= 2\mathbb{P}_0(B_1 \geq z(n-m)^{-1/2}) \\
&\leq 2\exp(-(1/2)z^2(n-m)^{-1}).
\end{aligned}$$

To continue onwards we are again going to require that

$$\frac{z^2x^2}{6m^2} - \frac{z^2+x^2}{2m} \le 0$$

where now $n - m \le z^2 \le m^{3/2}$. As $m > n/2$, we can use the reasoning from our previous argument to restrict ourselves to $3m \le z^2 \le m^{3/2}$. Rearranging we therefore require that

$$x^2 \le \frac{3mz^2}{z^2 - 3m}$$

and the right hand side is minimised when $z^2 = m^{3/2}$ and equals $3m^{3/2}/(\sqrt{m}-3)$. As $m > n/2$, for any fixed x we can pick n such that x^2 is smaller than this quantity, so the above is satisfied.

Using Corollary 3.7, Lemma 3.8 and substituting this in gives

$$\begin{aligned}
(3.15\,ii) &\lesssim \int_{\sqrt{n-m}}^{m^{3/4}} \frac{2zx}{m^{3/2}} \exp\left(\frac{z^2x^2}{6m^2} - \frac{z^2+x^2}{2m} - \frac{z^2}{2(n-m)}\right) \mathrm{d}z \\
&\le \int_0^\infty \frac{2zx}{m^{3/2}} e^{-z^2/(2(n-m))} \, \mathrm{d}z
\end{aligned} \tag{3.16}$$

and by Lemma 3.26, this is of order at most $(n-m)/n^{3/2}$. Again the implicit constant depends on x in a monotonically increasing fashion, which completes the proof of Proposition 3.17.

3.7 Proofs of Lemmas 3.13 and 3.14

We prove the final technical results needed here. The proof of Lemma 3.13 is almost identical to the proof of Lemma 2.9, we just have to work in terms of a Brownian motion coupled with a 2D PPP rather than a walk coupled with a sequence of 1D PPPs.

Proof of Lemma 3.13. Recall that because we are dealing with an intersection of decreasing events, we are allowed to intersect over $\mathbb{N}$ rather than $\mathbb{R}_+$. So as in the proof in the random walk case, due to how the T_n^α are nested, we have that

$$\Big(\bigcap_{n\ge1} \bar{T}_n^\alpha\Big) \setminus \Big(\bigcap_{n\ge1} T_n^\alpha\Big) \subset \bigcup_{N\ge1} \bigcap_{n\ge N} (\bar{T}_n^\alpha \setminus T_n^\alpha)$$

so it suffices to show that for each N

$$\bigcap_{n\geq N}(\bar{T}_n^\alpha \setminus T_n^\alpha) = \emptyset \quad \text{almost surely.}$$

Considering $\bar{T}_n^\alpha \setminus T_n^\alpha$ for fixed $n \in \mathbb{N}$, it is clear that the only points that can be in this set are the dynamical-times where a reflection occurs among the interval $[0, n]$ of Brownian time. That is, $\bar{T}_n^\alpha \setminus T_n^\alpha \subset \pi_T(R_{n,\infty})$.

$R_{n,\infty}$ has only countably many points almost surely; denote them by $(s_i, t_i) \in R_{n,\infty}$. As $B_{s_i}(t_i) - B_0(t_i)$ is independent of $B_m(t_i) - B_{s_i}(t_i)$ for all $m > s_i$, we have that at dynamical-time t_i there will exist Brownian-times after s_i where $B(t_i)$ hits 0 and $2B_{s_i}(t_i)$. This means that $B(t_i)$ and $B(t_i^-)$ both fall below $s \to s^\alpha$ for some s, where $B(t_i^-)$ is the Brownian path right before the reflection at time t_i. Thus $t_i \notin \bar{T}_n^\alpha \setminus T_n^\alpha$ for all large n, concluding the proof. □

The proof of Lemma 3.14 again follows the same outline as its random walk counterpart Lemma 2.10. However the probability space we need to set up to apply the ergodic theorem is slightly more involved to define. It turns out that the notation is less bothersome but the bits of notation we do have represent much more complicated objects. We will use the definitions provided in Section 2.7 and Theorem 2.22.

Proof of Lemma 3.14. We need to construct a σ-finite probability space to work with. Recall that to construct the process $(B_s(t))_{s,t\geq 0}$ we require $(B_s(0))_{s\geq 0}$ and a Poisson point process $\mathcal{P}$ of rate $1/2$ on $\mathbb{R}_+^2$.

We can create the non-dynamical process $(B_s(0))_{s\geq 0}$ via taking a sequence $(b_i)_{i\in\mathbb{N}}$ of independent Brownian motions on $[0, 1)$ and gluing the start point of b_2 to the end of b_1 (such that the result process is continuous) and then gluing on b_3 and so on. To be precise, these Brownian motions live in the probability space $(C_0([0, 1)), \mathcal{F}_b, \mathbb{P}_b)$ where $C_0([0, 1))$ is the set of real valued continuous functions on $[0, 1)$ that vanish at zero, $\mathbb{P}_b$ is the Wiener measure and $\mathcal{F}_b$ is the corresponding σ-algebra.

For the dynamics, we view each $R_i := ([i - 1, i) \times \mathbb{R}_+) \cap \mathcal{P}$ (where $i \in \mathbb{N}$) as a random measure rather than a random subset, which is the approach taken in [27]. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a probability space rich enough to construct a sequence of PPPs (R_i), which are measurable mappings from Ω' into $\mathbf{N}$ with respect to $\mathcal{N}$. $\mathbf{N}$ is the set of all measures ν that are the countable sum of measures ν^* on $[0, 1) \times \mathbb{R}_+$ such that $\nu^*(A) \in \mathbb{N}_0$ for all $A \in \mathcal{B}([0, 1) \times \mathbb{R}_+)$ ($\mathcal{B}$ for Borel-σ-algebra), while $\mathcal{N}$ is the σ-algebra on $\mathbf{N}$ generated by subsets of the form:

$$\{\nu \in \mathbf{N} \ : \ \nu(A) = k\}, \quad A \in \mathcal{B}([0,1) \times \mathbb{R}_+), \ k \in \mathbb{N}_0$$

Denoting by $\mathcal{M}$ the set of all measurable mappings $\mu : \Omega' \to \mathbf{N}$, we have a σ-algebra $\mathcal{M}^* = \sigma(X_{A'} \ : \ A' \in \mathcal{F}')$ on $\mathcal{M}$ where $X_{A'}(\mu) = \mu(A') \in \mathbf{N}$ for all $\mu \in \mathcal{M}$ and $A' \in \mathcal{F}'$.

Informally speaking, we can view $(R_i)_{i \in \mathbb{N}}$ as independent sets with the same law as $R_{1,\infty}$ and then stick them together side by side. The above construction can be easily understood if you think about it in reverse. We can simply chop our PPP$(1/2)$ on $\mathbb{R}^2_+$ into the union of rate $1/2$ PPPs on $R_{[i-1,i),\infty}$ for $i \in \mathbb{N}$, and we can take $(B_s(0))$ and "cut" it at Brownian times $1, 2, 3, \ldots$ to generate multiple length 1 Brownian motions.

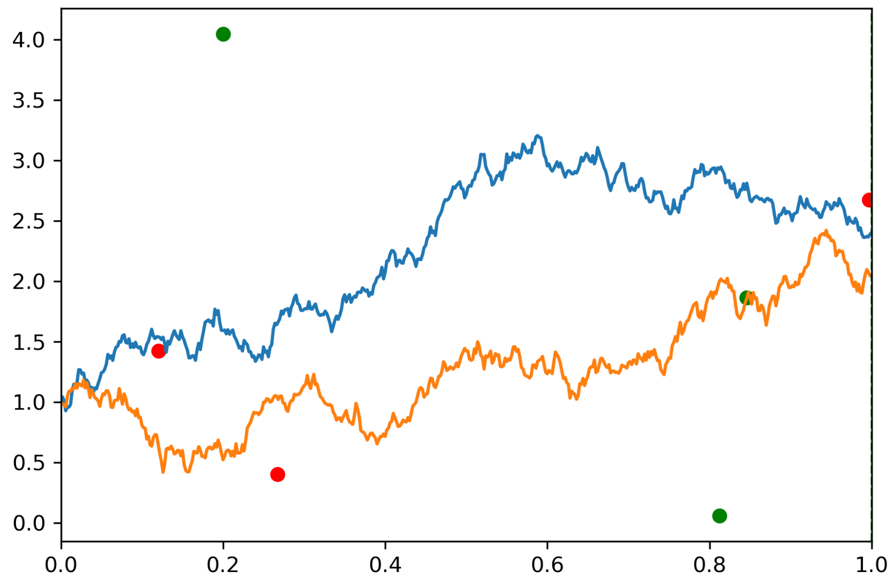

Figure 3-6: Two Brownian motions on $[0, 1)$ alongside Poisson points representing where/when reflections happen for each curve

Thus we can write the process $(B_s(t))_{s,t\geq 0}$ as $(b_i, R_i)_{i\in\mathbb{N}}$ and so define our space $\Omega = (C_0([0,1)) \times \mathcal{M})^{\mathbb{N}}$ as the set of sequences of this form, with the product σ-algebra $\mathcal{F}$ and the product measure μ defined so that under μ the b_i are Brownian motions and the R_i are PPPs. Since by gluing we can generate $(B_s(0))_{s\geq 0}$ and a PPP$(1/2)$ on $\mathbb{R}^2_+$, we can indeed build our dynamical Brownian motion using our definition.

We now define $\theta : \Omega \to \Omega$ by sending $(b_i, R_i)_{i\geq 1}$ to $(b_i, R_i)_{i\geq 2}$. This would be akin to deleting the blue Brownian motion and red Poisson points in Figures 3-6 and 3-7. Showing θ is ergodic is similar to how it is done in the random walk case. We define the

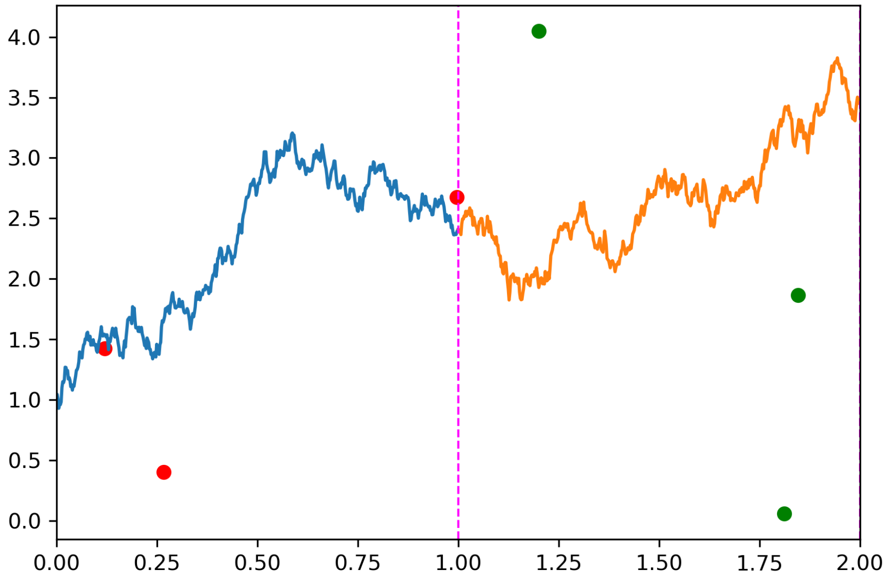

Figure 3-7: The two Brownian motions from Figure 3-6 glued together with the second set of Poisson points shifted by one. Cut along the magenta line to reverse this process.

set of cylinder sets $\mathcal{C}$ which form a π-system satisfying $\sigma(\mathcal{C}) = \mathcal{F}$. $\mathcal{C}$ consists of sets of the form $\bigotimes_{i,\omega} A_i^\omega$ where $i \in \mathbb{N}$ and $\omega \in \{b, R\}$. Every set of the form A_i^b is an element of $\mathcal{F}_b$ and all but finitely many of the (A_i^b) (over i) are equal to $C_0([0,1))$, while sets of the form A_i^R are elements of $\mathcal{M}^*$ where all but finitely many of the (A_i^R) (over i) are equal to $\mathcal{M}$. We also define the projection maps

$$\pi_i^b : \Omega \to C_0([0,1)),\ (b_i, R_i)_{i\geq 0} \mapsto b_i$$
$$\pi_i^R : \Omega \to \mathcal{M},\ (b_i, R_i)_{i\geq 0} \mapsto R_i$$

noting that $\mathcal{F}$ is the smallest σ-algebra making all of these maps measurable. For $A \in \mathcal{C}$, $\theta^{-1}(A) = (C_0([0,1)) \times \mathcal{M}) \times A \in \mathcal{C}$ so ω is measure-preserving (and indeed measurable). Define the tail σ-algebra $\mathcal{T}$ by

$$\mathcal{T} = \bigcap_{n\in\mathbb{N}} \mathcal{T}_n \text{ where } \mathcal{T}_n = \sigma(\pi_i^\omega \ :\ i > n,\ \omega \in \{b, R\}) \subset \mathcal{F}.$$

Now for $A \in \mathcal{C}$ (a cylinder set), we have that

$$\theta^{-n}(A) = \{x \in \Omega \ :\ \pi_{i+n}^\omega(x) \in A_i^\omega\ \forall i, \omega\} \in \mathcal{T}_n.$$

Now as $\mathcal{T}_n$ is a σ-algebra and $\mathcal{C}$ is a π-system, we have that $\theta^{-n}(A) \in \mathcal{T}_n\ \forall A \in \mathcal{F}$. Now if $A \in \mathcal{F}_\theta = \{A \in \mathcal{F} \ : \ \theta^{-1}(A) = A\}$, then $A = \theta^{-n}(A) \in \mathcal{T}_n$ for every n, so $A \in \mathcal{T}$, this immediately gives us that $\mathcal{F}_\theta \subset \mathcal{T}$, and so by Kolmogorov's $0-1$ law (see e.g. [5, Theorem 7.2.4]) we have that $\mathcal{T}$ is trivial, hence $\mathcal{F}_\theta$ is trivial, so θ is ergodic.

Define

$$\mathcal{E}'_\alpha = \Big\{t \in [0,1] : \liminf_{s\to\infty} \frac{-B_s(t)}{s^\alpha} > 0\Big\}.$$

For any $\alpha \geq 0$, the Hausdorff dimension of $\mathcal{E}_\alpha \cup \mathcal{E}'_\alpha$ is invariant under θ, and therefore constant almost surely by Theorem 2.22. By symmetry, the Hausdorff dimension of $\mathcal{E}_\alpha$ equals that of $\mathcal{E}'_\alpha$. Since the Hausdorff dimension of the union of two sets is the maximum of their Hausdorff dimensions, the Hausdorff dimension of $\mathcal{E}_\alpha$ must therefore equal that of $\mathcal{E}_\alpha \cup \mathcal{E}'_\alpha$, and thus be constant almost surely. □

3.8 Proofs of Theorem 3.15 and Corollary 3.16

In order to prove that the upper bound of the Hausdorff dimension of $\mathcal{E}_0$ was $1/2$, we needed to adapt theorems that utilise influences from a discrete setting to a continuous setting. To this end we stated Theorem 3.15 and Corollary 3.16, which we prove in this section.

Proof of Theorem 3.15. The proof is similar in flavour to the one given in [44, Theorem 8.1], just we need to be careful regarding continuous space. However, we have access to additional tools as our dynamical model is built via a Poisson point process.

For $n \in \mathbb{N}$ define $T_n := \{t \in [0,1] \ : \ B(t) \in A_n\}$ and $N_n = |\partial T_n|$, the (random) number of points on the boundary of T_n. We first prove that $\mathbb{E}[N_n] = (1/2)\mathcal{I}^1(A_n)$.

Mecke's equation [27, Theorem 4.1] for a Poisson process η with intensity λ is

$$\mathbb{E}\left[\int f(x,\eta)\eta(\mathrm{d}x)\right] = \int \mathbb{E}[f(x,\eta+\delta_x)]\lambda(\mathrm{d}x)$$

for all sufficiently nice functions f. For us, we just need it to hold for indicator functions, which it does. Here, $\eta = \tilde{\mathcal{P}} = \mathcal{P} \cap (\mathbb{R}_+ \times [0,1])$, i.e. a PPP(1/2) on $\mathbb{R}_+ \times [0,1]$. So that λ is $(1/2)Leb$ on $\mathbb{R}_+ \times [0,1]$. A point in $\tilde{\mathcal{P}}$ is written $x = (m,t)$.

Define $F = F((m,t), \tilde{\mathcal{P}}, (B_s(0)))$ as the event of "whether A_n occurs or not at dynamical time t changes if an extra point is added at (m,t), given all points in $\tilde{\mathcal{P}}$ and initial Brownian motion $(B_s(0))$", where A_n is the event from the theorem statement.

We define the function f in Mecke's equation as $\mathbb{E}[\mathbb{1}_F]$ where the expectation is taken with respect to the Brownian motion so that the randomness of the intial path is gone prior to applying Mecke's equation. The right-hand side of Mecke's equation is then

$$\begin{aligned}\frac{1}{2}\int_0^1\int_0^\infty &\mathbb{P}(F((m,t),\tilde{\mathcal{P}}+\delta_{(m,t)},(B_s(0))))\ \mathrm{d}m\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^1\int_0^\infty \mathbb{P}(F((m,t),\delta_{(m,t)},(B_s(0))))\ \mathrm{d}m\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^1\int_0^\infty \mathcal{I}_m(A_n)\ \mathrm{d}s\,\mathrm{d}t = \frac{1}{2}\mathcal{I}(A_n)\end{aligned}$$

where the first equality uses that reflected Brownian motions are Brownian motions themselves, and the second equality used that the dynamical time t the reflection occurs does not matter, just the positioning of it as it is the only reflection taking place.

For the left-hand side, the number of points in $\tilde{\mathcal{P}}$ is countable a.s., so the integral can be written as a sum as follows:

$$\mathbb{E}\left[\sum_{(m,t)\in\tilde{\mathcal{P}}}\mathbb{E}[\mathbb{1}_{F((m,t),\tilde{\mathcal{P}},(B_s(0)))}]\right] = \mathbb{E}[N_n].$$

The internal expectation is again with respect to the Brownian motion. The equality holds as N_n is precisely the number of dynamical times where A_n switches from occurring to not occurring or vice-versa, and each such dynamical time corresponds to a time where F occurs.

The rest of the proof follows similarly to the proof of [44, Theorem 8.1], but we shall write it here in full for completeness.

For $\varepsilon > 0$ let $\mathcal{N}(U,\varepsilon)$ be the number of ε-intervals needed to cover U. Also write $|I|$ is the Lebesgue measure of I. By definition of N_n, T_n consists of $N_n/2$ intervals of combined length $|T_n|$. For an interval I, it takes at most $(|I|/\varepsilon + 1)$ intervals of length ε to cover I. Combining these facts, we have that

$$\mathcal{N}(T_n,\varepsilon) \le \frac{1}{\varepsilon}|T_n| + \frac{1}{2}N_n$$

which, using Fubini's theorem and our formula for $\mathbb{E}[N_n]$, gives

$$\mathbb{E}[\mathcal{N}(T_n,\varepsilon)] \le \frac{1}{\varepsilon}\mathbb{E}[|T_n|] + \frac{1}{2}\mathbb{E}[N_n] \le \frac{1}{\varepsilon}\mathbb{P}(A_n) + \frac{1}{4}\mathcal{I}^1(A_n).$$

Now by assumption $\liminf_n \mathcal{I}^1(A_n) = \infty$, so that $a_n := \mathbb{P}(A_n)/\mathcal{I}^1(A_n) \to 0$ and

$$\mathbb{E}[\mathcal{N}(T, a_n)] \leq \mathbb{E}[\mathcal{N}(T_n, a_n)] \leq \frac{5}{4}\mathcal{I}^1(A_n). \tag{3.17}$$

By passing to a subsequence if necessary, we can change the liminf in the theorem statement to a lim and call it L. By definition of L, $\forall \varepsilon > 0$ and sufficiently large n we have

$$1 - \frac{1}{L+\varepsilon} \geq \frac{\log \mathbb{P}(A_n)}{\log \mathcal{I}^1(A_n)}$$

which implies

$$\mathcal{I}^1(A_n)^{1-(L+\varepsilon)^{-1}} \geq \mathcal{I}^1(A_n)^{\frac{\log \mathbb{P}(A_n)}{\log \mathcal{I}^1(A_n)}} = e^{\log \mathbb{P}(A_n)} = \mathbb{P}(A_n).$$

Rearranging this inequality gives us that

$$\mathcal{I}^1(A_n) \leq \left(\frac{\mathcal{I}^1(A_n)}{\mathbb{P}(A_n)}\right)^{L+\varepsilon} = a_n^{-(L+\varepsilon)}$$

which combined with (3.17), means that for any $\varepsilon > 0$

$$\mathbb{E}[\mathcal{N}(T, a_n)] \leq \frac{5}{4}\mathcal{I}^1(A_n) = \frac{5}{4}a_n^{-(L+\varepsilon)}. \tag{3.18}$$

As $a_n = \mathbb{P}(A_n)/\mathcal{I}(A_n) \to 0$, we can pass to a subsequence (b_n) of (a_n) such that $b_n \leq n^{-2\varepsilon^{-1}}$. Our Hausdorff content of T for dimension $d \geq 0$ is (see Section 1.1.3 for a reminder of the definition)

$$\mathcal{H}^d_{b_n}(T) \leq \mathcal{N}(T, b_n) b_n^d$$

as there are $\mathcal{N}(T, b_n)$ intervals by definition, each of width b_n. The Hausdorff dimension of T is the infimum of all d such that

$$\mathcal{H}^d_{b_n}(T) \to 0 \ a.s.$$

By the Borel-Cantelli lemma, given d we have $\mathcal{H}^d(T) = 0$ if for all $\delta > 0$

$$\sum_{n=1}^{\infty} \mathbb{P}(\mathcal{N}(T, b_n) b_n^d \geq \delta) < \infty.$$

Set $d = L + 2\varepsilon$. By Markov's inequality and (3.18), we have

$$\sum_{n=1}^{\infty} \mathbb{P}(\mathcal{N}(T, b_n) b_n^{L+2\varepsilon} \geq \delta) \leq \frac{5}{4\delta} \sum_{n=1}^{\infty} b_n^{\varepsilon} \lesssim \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Therefore, given any $\varepsilon > 0$, the Hausdorff dimension of T is $\leq L + 2\varepsilon$ almost surely. This gives the required upper bound on the Hausdorff dimension. $\square$

The proof of Corollary 3.16 is short, we just need to show that the hypotheses of Theorem 3.15 apply for $A_n = P_{k,l,n}$.

Proof of Corollary 3.16. It is clear that if $A_n = P_{k,l,n}$ then $T = T'_{k,l}$. $P_{k,l,n}$ only depends on the Brownian motion up until Brownian time $k + n$, which is finite. Also by Corollary 3.11

$$\mathbb{P}(P_{k,l,n}) \leq \mathbb{P}_{l+1}(P_n) \asymp n^{-1/2} \to 0.$$

We finally prove that $\liminf_{n\to\infty} \mathcal{I}^1(P_{k,l,n}) = \infty$ as

$$\begin{aligned} \mathcal{I}^1(P_{k,l,n}) = \int_0^{\infty} \mathcal{I}^1_m(P_{k,l,n})\,\mathrm{d}m \geq \int_k^{k+n} \mathcal{I}^1_m(P_{k,l,n})\,\mathrm{d}m \\ \geq \mathbb{P}_1(B_k \in [l, l+1)) \int_0^n \inf_{x\in[l,l+1)} \mathcal{I}^x_m(P_n)\,\mathrm{d}m \\ \gtrsim \mathbb{P}_1(B_k \in [l, l+1)) \int_0^n \frac{n-m}{n^{3/2}}\,\mathrm{d}m \to \infty \end{aligned}$$

where we have used Proposition 3.17.

$\square$

3.9 Proof of Lemma 3.18

This proof is significantly more involved than the random walk equivalent, Lemma 2.13, as we use some somewhat detailed stochastic calculus. Recall that

$$P_r = \{\mathcal{B}_s > 0 \ \ \forall 0 < s \leq r\} \quad \text{and} \quad P_r^{\alpha} = \big\{\mathcal{B}_s > (s+1)^{\alpha} - 1 \ \ \forall 0 < s \leq r\big\},$$

We prove Lemma 3.18 by using some sublemmas which we will also prove. The first of which is a Lemma that controls the probability that $\mathcal{B}$ stays above 0 but equally doesn't grow too fast:

Lemma 3.27. *Fix $\beta \in (1/2, 1]$, and for any $A > 0$ define $T = T(A, \beta) = \inf\{s > 0$:*

$B_s \geq A(s+1)^\beta\}$. *Then* $\exists A = A(\beta) > 0$ *such that for all* $r > 0$

$$\mathbb{P}(P_r,\ T > r) \geq \frac{1}{2}\mathbb{P}(P_r).$$

The next Lemma relates the probability we are after into one of the form of Lemma 3.27:

Lemma 3.28. *For any* $\alpha \in [0, 1/2)$ *there exists* $\beta = \beta(\alpha) \in (1/2, 1]$, $A = A(\beta(\alpha)) > 0$ *(A depends on* α *through the choice of* β*) and an independent constant* $1 > c > 0$ *such that for all* $r > 0$

$$\mathbb{P}(P_r^\alpha) \geq c\mathbb{P}(P_r,\ T > r).$$

Proof of Lemma 3.18. Combining Lemmas 3.27 and 3.28 (with A large enough so that both hold) we have that $\mathbb{P}(P_r^\alpha) \geq (c/2)\mathbb{P}(P_r)$. It is also clear that $\mathbb{P}(P_r^\alpha) \leq \mathbb{P}(P_r)$, so it suffices to prove that $\mathbb{P}(P_r) \asymp r^{-1/2}$. This can be seen from Corollary 3.11. $\square$

We next prove Lemma 3.27, it isn't too demanding technically but there are a lot of computations and bounds that need to be used.

Proof of Lemma 3.27. Take $\beta \in (1/2, 1]$, $A > 0$ and T as in the statement. Now

$$\begin{aligned}
\mathbb{P}(P_r,\ T \leq r) &\leq \mathbb{P}(P_r,\ T \leq \lceil r \rceil) \\
&= \sum_{j=1}^{\lceil r \rceil} \mathbb{P}(P_r,\ T \in [j-1, j]) \\
&\leq \sum_{j=1}^{\lceil r \rceil} \mathbb{P}(T \in [j-1, j])\mathbb{P}(P_r \mid T \in [j-1, j]) \\
&\leq \sum_{j=1}^{\lceil r \rceil} \mathbb{P}(T \in [j-1, j])\mathbb{P}(B_s > -1\ \forall j \leq s \leq r \mid T \in [j-1, j]) \\
&\leq \sum_{j=1}^{\lceil r \rceil} \mathbb{P}\left(\sup_{s \leq j} B_s \geq Aj^\beta\right)\mathbb{P}_{A(j+1)^\beta}(P_{r-j}) \qquad (3.19)
\end{aligned}$$

where when $j = \lceil r \rceil$ we view the second probability as 1, else we would be asking a Brownian motion to stay positive for a negative amount of Brownian time. As we will shortly see, we will in fact bound this probability above by one for all $j > \lfloor r/2 \rfloor$. In the above computation we have used that $T \in [j-1, j]$ implies the supremum of B_s

over that interval (and thus over $[0, j]$) must be at least Aj^β. Also, $\mathbb{P}(B_s > -1\ \forall j \le s \le r \mid T \in [j-1, j]) \le \mathbb{P}_{A(j+1)^\beta}(P_{r-j})$ by the strong Markov property.

We now simplify the probabilities involved in the sum, for the first probability (involving the supremum), we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{s \le j} B_s \ge Aj^\beta\right) &= \mathbb{P}(|B_j| \ge Aj^\beta) \\
&= 2\mathbb{P}\left(B_1 > Aj^{\beta - 1/2}\right) \\
&\le \frac{2\exp(-\frac{1}{2}A^2 j^{2\beta-1})}{A\sqrt{2\pi}j^{\beta-1/2}} \qquad (3.20)
\end{aligned}$$

where the first equality is from Lemma 3.10, and from there we use Brownian scaling and the fact that $B_1 \sim N(0,1)$. To be more precise ;

$$\begin{aligned}
\mathbb{P}(B_1 > a) = \frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-x^2/2}\,\mathrm{d}x &< \frac{1}{\sqrt{2\pi}}\int_a^\infty \frac{x}{a}e^{-x^2/2}\,\mathrm{d}x \\
&= \frac{1}{a\sqrt{2\pi}}\int_{a^2/2}^\infty e^{-z}\,\mathrm{d}z = \frac{1}{a\sqrt{2\pi}}e^{-\frac{a^2}{2}}.
\end{aligned}$$

Moving onto $\mathbb{P}_{A(j+1)^\beta}(P_{r-j})$, we have for $j < \lceil r \rceil$ that (for $j = \lceil r \rceil$ we bound the probability by 1):

$$\begin{aligned}
\mathbb{P}_{A(j+1)^\beta}(P_{r-j}) &= \mathbb{P}(B_s > -(1 + A(j+1)^\beta)\ \forall s \le r-j) \\
&= \mathbb{P}\left(\inf_{s \le r-j} B_s > -(1 + A(j+1)^\beta)\right) \\
&= \mathbb{P}\left(|B_1| \le \frac{1 + A(j+1)^\beta}{\sqrt{r-j}}\right) \le \frac{2(1 + A(j+1)^\beta)}{\sqrt{2\pi(r-j)}} \wedge 1. \qquad (3.21)
\end{aligned}$$

Before plugging these two in, we're going to split the sum in (3.19) into two depending on whether $j \le \lfloor r/2 \rfloor$ or not. In the former case, we use the left hand portion of the wedge in (3.21) to say that (for a constant C)

$$(3.21) \le C \times \frac{A(j+1)^\beta}{\sqrt{r}}$$

so that our upper bound for (3.22) for these values of j is

$$C\frac{2A(j+1)^\beta \exp(-\frac{1}{2}A^2 j^{2\beta-1})}{A\sqrt{2r\pi}j^{\beta-1/2}} \le \frac{C}{\sqrt{r}}\sqrt{j}\exp\left(-\frac{1}{2}A^2 j^{2\beta-1}\right).$$

Note we have used that $((j+1)/j)^\beta \le 2$ as $\beta \le 1$, and then absorbed constants into C.

We move on to the case where $j > \lfloor r/2 \rfloor$. Here we use the right hand part of the wedge in (3.21), so that we just need to bound (3.20). This means we have a sum of at most $r/2$ terms where each term is at most (for a constant C')

$$C' \times \frac{r^{1/2}}{A} \exp\left(-\frac{A^2}{2}\left(\frac{r}{2}-1\right)^{2\beta-1}\right).$$

To summarise, for some constants C'' and C''':

$$(3.19) \le \frac{C''}{\sqrt{r}} \sum_{j=1}^{\lceil r/2 \rceil} j^{1/2} \exp\left(-\frac{1}{2}A^2 j^{2\beta-1}\right) + C''' \frac{r^{3/2}}{A} \exp\left(-\frac{A^2}{2}\left(\frac{r}{2}-1\right)^{2\beta-1}\right). \quad (3.22)$$

Given $\beta > 1/2$ we know that for any $A > 0$, $C'' r^{3/2} A^{-1} \exp(-(A^2/2)((r/2)-1)^{2\beta-1}) \to 0$ as $r \to \infty$, and is also decreasing in A. In particular this convergence is always faster than $r^{-1/2}$. Thus there exists sufficiently large $A_1 = A_1(\beta) > 0$ such that this second term is $\le (1/2)(2er\pi)^{-1/2} \le (1/4)\mathbb{P}(P_r)$. Similarly the first term in the above is less than a constant times $r^{-1/2}$ as

$$\sum_{j=1}^{\lceil r/2 \rceil} j^{1/2} \exp\left(-\frac{1}{2}A^2 j^{2\beta-1}\right) \le \sum_{j=1}^{\infty} j^{1/2} \exp\left(-\frac{1}{2}A^2 j^{2\beta-1}\right) = K$$

where $K = K(A)$ is a constant that depends on A. It is clear that $K(A)$ decreases to 0 as A increases, so again there exists $A_2 = A_2(\beta)$ such that the first term is $\le (1/4)\mathbb{P}(P_r)$.

To conclude, given $\beta > 1/2$ there exists $A(\beta) = \max\{A_1(\beta), A_2(\beta)\} > 0$ such that (3.22) is less than or equal to $(1/2)\mathbb{P}(P_r)$. So for sufficiently large A

$$\mathbb{P}(P_r,\ T > r) \ge \frac{1}{2}\mathbb{P}(P_r). \qquad \square$$

To prove Lemma 3.28 we utilise Girsanov's Theorem with a change of a measure and results from stochastic calculus. For a general overview of this material we suggest Steele's book [47].

Proof of Lemma 3.28. By [47, Theorem 13.4] we have that for any continuously differentiable (non-random) function f,

$$\exp\left(\int_0^t f'(s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t f'(s)^2\,\mathrm{d}s\right)$$

is a non-negative martingale with expectation one, provided that it satisfies the Novikov condition:

$$\exp\left(\frac{1}{2}\int_0^t f'(s)^2\,\mathrm{d}s\right) < \infty.$$

If f'' also exists then by stochastic integration by parts we also have that

$$d(f'(s)B_s) = f'(s)\,\mathrm{d}B_s + f''(s)B_s\,\mathrm{d}s$$

meaning that

$$\int_0^t f'(s)\,\mathrm{d}Bs = f'(t)B_t - f'(0)B_0 - \int_0^t f''(s)B_s\,\mathrm{d}s \tag{3.23}$$

(note that we have $B_0 = 0$). Now we define a new probability measure $\mathbb{Q}$ via the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_r} = \exp\left(\int_0^r f'(s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^r f'(s)^2\,\mathrm{d}s\right)$$

and then by Girsanov's Theorem [47, Theorem 3.12] (as well as remarks preceding [47, Theorem 3.14]) we have that $(B_s - f(s))_{s\geq 0}$ is a Brownian motion under $\mathbb{Q}$. Let $f(s) = (s+1)^\alpha - 1$ for some $\alpha \in [0, 1/2)$. It is clear that the Novikov condition applies. Now

$$\mathbb{P}(P_r^\alpha) = \mathbb{P}(B_s > f(s) - 1\ \forall s \leq r) \geq \mathbb{P}(E_f)$$

where for a map g the event E_g is

$$E_g := \{g(s) - 1 < B_s \leq A(s+1)^\beta + g(s)\ \forall s \leq r\}$$

and E_0 is E_g for $g = 0$, the zero map. Note that $E_0 = \{P_r,\ T > r\}$. Now,

$$
\begin{aligned}
\mathbb{P}(E_f) &= \mathbb{E}_{\mathbb{Q}}[\exp(-\int_0^r f'(s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^r f'(s)^2\,\mathrm{d}s)\mathbb{1}_{E_f}] \\
&= \mathbb{E}_{\mathbb{Q}}[\exp(-f'(r)B_r + f'(0)B_0 + \int_0^t f''(s)B_s\,\mathrm{d}s + \frac{1}{2}\int_0^r f'(s)^2\,\mathrm{d}s)\mathbb{1}_{E_f}] \\
&= \mathbb{E}_{\mathbb{P}}[\exp(-f'(r)(B_r + f(r)) + \int_0^r f''(s)(B_s + f(s))\,\mathrm{d}s + \frac{1}{2}\int_0^r f'(s)^2\,\mathrm{d}s)\mathbb{1}_{E_0}] \\
&\geq \mathbb{P}(E_0)\exp\Big(-f'(r)(A(r+1)^\beta + f(r)) \\
&\qquad\qquad + \int_0^r f''(s)(A(s+1)^\beta + f(s))\,\mathrm{d}s + \frac{1}{2}\int_0^r f'(s)^2\,\mathrm{d}s\Big). \qquad (3.24)
\end{aligned}
$$

The second equality is due to (3.23), while third equality comes from the fact that (B_s) under $\mathbb{Q}$ has the same law as $(B_s + f(s))$ under $\mathbb{P}$. The inequality comes from replacing B_s with $A(s+1)^\beta$ which minimises the entire term given E_0. This is because our function f satisfies (for $s \geq 0$) $f(s) = (s+1)^\alpha - 1 \geq 0$, $f'(s) = \alpha(s+1)^{\alpha-1} > 0$ and $f''(s) = -\alpha(1-\alpha)(s+1)^{\alpha-2} < 0$, and also on E_0 we have $-1 \leq B_s \leq A(s+1)^\beta$. So to minimise the whole exponential we need to maximise (B_s) as to maximise the negative sign of the first two terms in the exponent, so we set $B_s = A(s+1)^\beta$.

We have that $\alpha < 1/2$ and pick $\beta > 1/2$ such that $\alpha + \beta < 1$ (e.g. $\beta = (3/4) - (\alpha/2)$). We now substitute $f(s)$, $f'(s)$ and $f''(s)$ as stated above into (3.24). This leads to the following simplifications

$$
\begin{aligned}
-f'(r)(A(r+1)^\beta + f(r)) &= -A\alpha(r+1)^{\alpha+\beta-1} - \alpha(r+1)^{2\alpha-1} + \alpha(r+1)^{\alpha-1} \\
\frac{1}{2}\int_0^r (f'(s))^2\ \mathrm{d}s &= -\frac{\alpha^2(r+1)^{2\alpha-1}}{2(1-2\alpha)} + \frac{\alpha^2}{2(1-2\alpha)} \\
\int_0^r f''(s)(A(s+1)^\beta + f(s))\ \mathrm{d}s &= \frac{A\alpha(1-\alpha)(r+1)^{\alpha+\beta-1}}{1-(\alpha+\beta)} + \frac{\alpha(1-\alpha)(r+1)^{2\alpha-1}}{1-2\alpha} \\
&\quad - \alpha(r+1)^{\alpha-1} + \alpha - \frac{\alpha(1-\alpha)}{1-2\alpha} - \frac{A\alpha(1-\alpha)}{1-(\alpha+\beta)}.
\end{aligned}
$$

Combining the above gives that the exponential term in (3.24) is

$$
\exp\left(\frac{A\alpha\beta(r+1)^{\alpha+\beta-1}}{1-(\alpha+\beta)} + \frac{\alpha^2(r+1)^{2\alpha-1}}{2(1-2\alpha)} + \alpha - \frac{\alpha(2-3\alpha)}{2(1-2\alpha)} - \frac{A\alpha(1-\alpha)}{1-(\alpha+\beta)}\right)
$$

which is decreasing and, as all the powers of $r+1$ are negative, approaches a non-zero constant $\exp(-C)$ as $t \to \infty$, thus is bounded below by $c = \exp(-C)$ for all r. So we have that $\mathbb{P}(P_r^\alpha) \geq c\mathbb{P}(E_0) = c\mathbb{P}(P_r,\ T > r)$. $\square$

Chapter 4

Dynamical Branching Random Walks

In this chapter, we build on the specific example of a branching random walk model studied by Bramson [11]. This example was discussed in Section 1.6 and we now introduce dynamics as in Section 1.1.3.

The object of interest in Bramson's work was M_n, the position of the left-most particle after n generations. As seen in Theorem 1.10, as n grows large, M_n grows like $(1/\log 2)\log\log n$. In this chapter, we investigate the hypothesis that there almost surely exist exceptional times where the left-most particle diverges to infinity slower than $(1/\log 2)\log\log n$. We also study bounds on probabilities of the form $\mathbb{P}(M_n \leq m(n,\gamma))$ for $m(n,\gamma) = (1/\log\gamma)\log\log n$ and $\gamma > 2$.

This chapter covers work in progress, ergo our main results will be presented as conjectures rather than as theorems. However, a lot of interesting mathematics has already been proven and will be presented here. We shall make clear why we believe these conjectures to be true, and what still needs to be done to complete the proofs.

4.1 Introduction

4.1.1 The dynamical model and conjectures

We begin by reminding ourselves of the static branching random walk (BRW) model, which was fully defined in Section 1.6. Our walk is defined on $\mathbb{N}_0$ by saying that in any given generation, every particle currently alive has two children (and then dies) where each child independently decides with probability 1/2 whether or not to jump one position to the right. In generation zero we have a single particle positioned at zero.

To be precise, our model is defined on the infinite binary tree, a Galton-Watson tree with offspring distribution $L = 2$. Every edge is equipped with an independent random variable with distribution X, where

$$\mathbb{P}(X = 1) = \mathbb{P}(X = 0) = 1/2.$$

We denote the root r, and define the node $r_{i_1 \dots i_n}$ (where each $i_k \in \{0, 1\}$) to be the $(i_1 + 1)$th child of r, the $(i_2 + 1)$th child of r_{i_1}, etc. We call the edge variable connecting from $r_{i_1 \dots i_{n-1}}$ to $r_{i_1 \dots i_n}$ $X(r_{i_1 \dots i_n})$ (or $X_{r_{i_1 \dots i_n}}$). The position of $r_{i_1 \dots i_n}$ in the BRW is given by

$$S_{r_{i_1 \cdots n}} = \sum_{k=1}^{n} X_{r_{i_1 \dots i_k}},$$

the sum of the values of the edges leading from r to $r_{i_1 \dots i_n}$. The quantity of interest is

$$M_n = \min_{i_1, \dots, i_n} S_{r_{i_1 \dots i_n}},$$

the position of the left-most particle after n generations. Figure 1-10 back in Section 1.6 gives an example of the first few generations of a branching random walk. We now introduce the dynamical model as per the method in Section 1.1.3. Denoting the vertex set of our binary tree by V, for every $v \in V$, except $v = r$ which has no edge leading to it, let $(N_v(t))_{t \geq 0}$ be an independent Poisson process of rate 1. Also for all $i \in \mathbb{N}_0$, define X_v^i to be an independent random variable with the same law as X above. Now define

$$X_v(t) = X_v^i \text{ whenever } N_v(t) = i.$$

This then gives us a dynamical branching random walk as for any time t we can define

$$S_{r_{i_1 \dots i_n}}(t) = \sum_{k=1}^{n} X_{r_{i_1 \dots i_k}}(t)$$

and

$$M_n(t) = \min_{i_1, \dots, i_n} S_{r_{i_1 \dots i_n}}(t).$$

It is worth reemphasising that the random variable $X_v(t)$ is attached to the edge leading to v from below, it is not attached to the vertex v itself.

Note that if we are looking at the process up to generation n then there are $2+4+\cdots+2^n = 2(2^n-1)$ exponential clocks running at once, as parents from previous generations can change their mind about whether they jump right or not.

Recall that Bramson [11] has proven that

$$M_n(0) \sim \frac{1}{\log(2)} \log\log(n) \ a.s.$$

where $f(n) \sim g(n)$ means that $f(n)/g(n) \to 1$ as $n \to \infty$. We conjecture that almost surely there are exceptional times where $M_n(t)$ grows slower than expected by a constant factor.

Heuristically we don't expect there to be times where $M_n(t)$ diverges faster than $(1/\log(2))\log\log(n)$ because for this to happen, we need every generation n particle (in the limit) to be pushed further to the right together, and given how many clocks there are this seems unlikely. Thus we expect

$$\sup_{t\in[0,\infty)} M_n(t) \sim \frac{1}{\log(2)} \log\log(n) \ a.s.$$

However the infimum is more interesting, as to reduce the minimum displacement it suffices to have a single unlucky particle, which is feasible given how many particles there are at generation n. With this in mind, we have the following conjecture:

Conjecture 4.1. *For all $\gamma > 2$ there almost surely exist exceptional times t where*

$$\liminf_{n\to\infty} \frac{\log(\gamma)M_n(t)}{\log\log(n)} \leq 1.$$

Throughout this chapter we define

$$m = m(n,\gamma) := \frac{1}{\log(\gamma)} \log\log(n).$$

In particular, m is a function of n and γ, despite the fact we hide this detail in the notation. The above conjecture therefore states that almost surely there is an exceptional time where $\liminf_n m^{-1}M_n(t) \leq 1$.

We also conjecture a deviation result for M_n:

Conjecture 4.2. *Let $\gamma > 2$, we have that (w.r.t n)*

$$\mathbb{P}(M_n \leq m(n,\gamma)) \asymp \exp\left(-\left(\frac{\gamma}{2}\right)^{m(n,\gamma)}\right) = n^{-2^{-m(n,\gamma)}}.$$

We can prove that $\mathbb{P}(M_n \geq m)$ is at least $n^{-2^{-m}}$ but we do not currently have a proof for the upper bound. Luckily, the lower bound is the more insightful of the two directions. The proof of the lower bound is in Section 4.7.

Note: Throughout this chapter, if there's a value e.g. m, which may not be an integer, but from context must be, then we mean the floor of that value. Due to the work in this chapter being a work in progress, these sorts of minor technicalities are not dealt with here.

4.1.2 Chapter layout

The rest of the chapter is structured as follows; in Section 4.2 we introduce the dynasty interpretation of our branching random walk which was used by Bramson, which is key for our proofs. We also deal with all the static (non-dynamical) results related to dynasties that we will need, which requires working with critical Galton-Watson processes. None of these results are new, but we have not found them explicitly written in the literature. In Section 4.3 we sketch what we believe will form the proof of Conjecture 4.1, as well as give a more detailed outline of what we believe the proof will look like, including results we need to prove. As proving exceptional times exist requires a second moment argument, Section 4.4 deals with the lower bound on the first moment that we need, which we have proved rigorously. Section 4.5 discusses most of the mathematics that is needed to turn Conjecture 4.1 into a theorem. We prove results regarding the dynamical branching random walk that we believe will be useful for getting an accurate second moment bound, and explain what is currently missing from our proof. As we will see, the event we hope to use the second moment method on is not the same as the event we need. Section 4.6 deals with this as well as other technicalities that we will need to resolve. Finally, Section 4.7 covers all our work on Conjecture 4.2 and explains why the result, if true, is interesting.

4.2 Dynasties and critical Galton-Watson forests

We start off by defining a dynasty and how to construct a branching random walk from a collection of dynasties, via an offspring distribution L. The following subsection defines the forward edge boundary, which provides additional rigour for our dynasty construction. The last subsection proves a result regarding the Galton-Watson process generated by L, allowing us to analyse how large individual dynasties can be. Nothing in this section is new, but we could not find explicit references for the lemmas we prove.

4.2.1 Dynasties

When studying the non-dynamical branching random walk, or equally the branching random walk at a fixed time t then the concept of dynasties comes into play, and is a useful way of viewing the process.

We say that a particular node $r_{i_1\ldots i_n}$ is located in the kth dynasty (for $k \in \mathbb{N}_0$) if $S(r_{i_1\ldots i_n}) = k$. In other words, if that node is distance k from the root with respect to the random walk jumps. Note that a parent and a child node are in the same dynasty iff the edge between them is zero valued. That is, edges that are valued one cause a jump from one dynasty to another.

We can also interpret this from the lens of percolation, in that each tree in a particular dynasty is a connected 0-component of vertices. This means that if we delete every edge valued one, we break our binary tree down into disjoint subtrees where each node in each subtree is in the same dynasty. In particular, the component containing the root consists of precisely all the particles that are in the 0th dynasty. However for the 1st, 2nd, … dynasties we have multiple components. See Figures 4-1 and 4-2 for this to be shown clearly.

If all we care about are the sizes and shapes of the dynasties, we can use the following construction instead, which forgets the shape of the initial binary tree. Consider the Branching Process $(Z_n)_{n\geq 0}$ given by $Z_0 = z_0$ and offspring distribution L where

$$\mathbb{P}(L=0) = \frac{1}{2}\mathbb{P}(L=1) = \mathbb{P}(L=2) = \frac{1}{4}.$$

In other words, the PGF of L is given by $g(s) = \frac{1}{4} + \frac{1}{2}s + \frac{1}{4}s^2$. Now:

1. Let $(Z_n^{(0)})$ be a Galton-Watson tree with the same law as (Z_n) with $Z_0 = 1$. Position this tree at zero (i.e. it is the 0th dynasty).

2. Recursively define, for $i \in \mathbb{N}$ (i.e. the ith dynasty), $(Z_n^{(i)})$, a Galton-Watson forest with the same law as (Z_n) but with $Z_0^{(i)} = \mathcal{P}^{(i-1)} + Z_0^{(i-1)}$, where

$$\mathcal{P}^{(i-1)} := \sum_{n=0}^{\infty} Z_n^{(i-1)}.$$

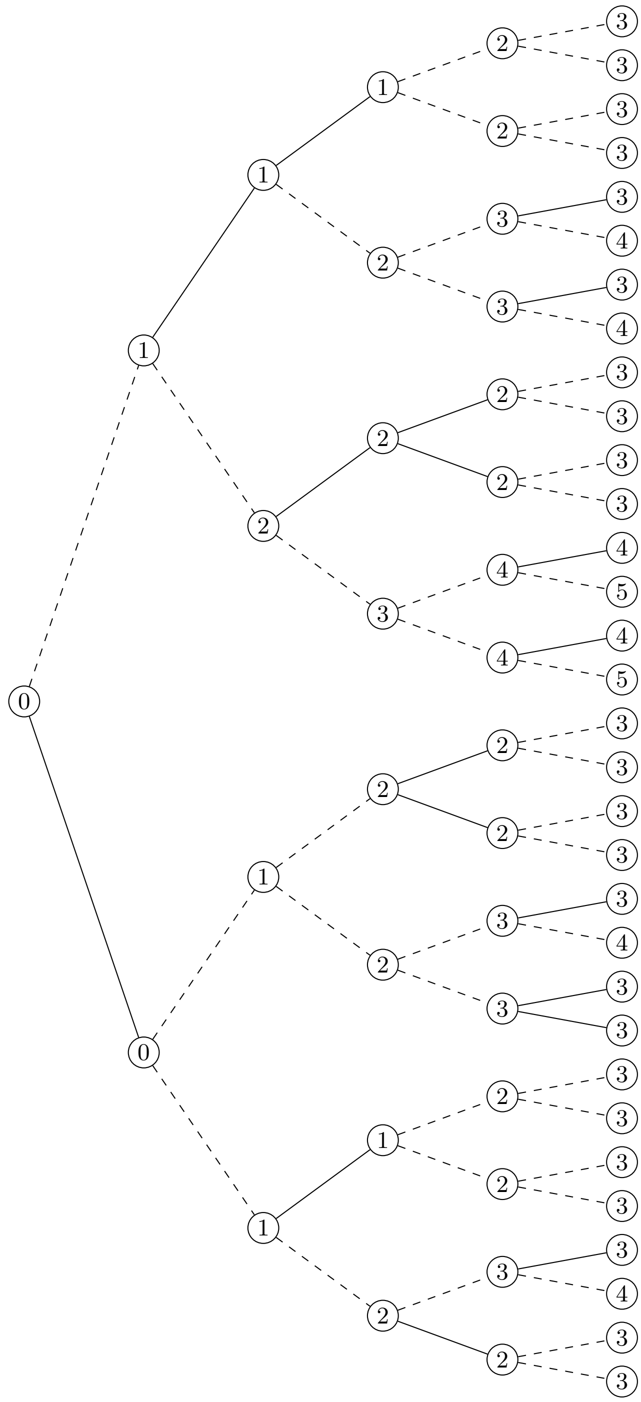

Figure 4-1: A realisation of the branching random walk up until the fifth generation. Values inside the nodes are S_v where v is that particular node. solid edges are zero-values while dotted edges are equal to one.

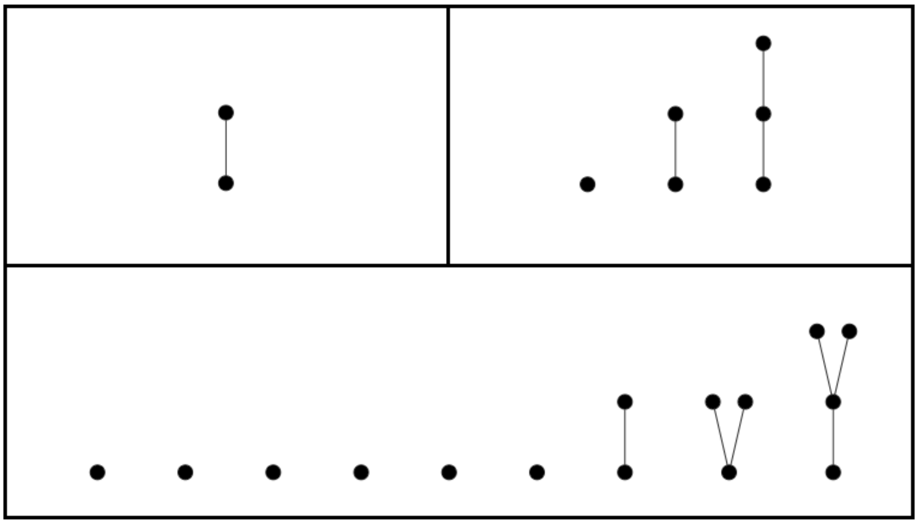

Figure 4-2: The 0th, 1st and 2nd dynasties from Figure 4-1 are in the top-left, top-right and bottom panels respectively. The ordering of the individual trees is unimportant, but every root (within a dynasty) being at the same height is vital.

That is $\mathcal{P}^{(i-1)}$ is the total progeny of the $(i-1)$th dynasty.

3. We denote by $F^{(i)}$ the entire forest generated by $(Z_n^{(i)})$, that is, the information regarding the shapes of individual trees, such as which node is parent to whom. Also for fixed time t, we define e.g. $Z_n^{(i)}(t)$ for the nth generation of the ith dynasty and time t.

The above construction is equivalent to taking our binary tree with labelled edges, deleting all edges valued one, and then grouping components belonging to dynasty i together, ensuring that roots of every component are of the same height.

This construction loses information regarding what happens at generation n of the original binary tree, since we generate $F^{(0)}, F^{(1)}, \dots$ in order, rather than going up through the initial binary tree generation by generation. However, a key fact we do use can still be seen in this setup, which is that if $Z_n^{(m)} > 0$ then $M_n \leq m$.

4.2.2 The forward edge boundary

We define the forward edge boundary of a subtree S of a tree T, with vertex and edge sets (V_S, E_S) and (V_T, E_T) respectively, as

$$\{e \in E_T \backslash E_S \ : \exists v_1 \in V_S, v_2 \in V_T, \ Gen(v_2) > Gen(v_1), \ e \text{ connects } v_1 \text{ and } v_2 \text{ in } T\}.$$

In other words, the forward edge boundary consists of all edges surrounding the subtree, that point upwards to a later generation.

Proposition 4.3. *A subtree consisting of $k \in \mathbb{N}$ vertices of the (infinite) binary tree has a forward edge boundary consisting of $k+1$ edges.*

Proof. This is a simple proof by induction. For the base case, $k = 1$, we simply have the root vertex, so the forward edge boundary consists of the two edges connected to the root. Now we assume that any subtree with k (fixed) vertices has a forward edge boundary of size $k + 1$.

Consider a subtree T with $k + 1$ vertices and label one of its leaves l. Denote by T' the tree of T with l and its connecting edge deleted. T' consists of k vertices so by assumption must have a size $k + 1$ forward edge boundary. One of these $k + 1$ edges on the boundary is the edge connecting l to T', so doesn't contribute to the forward edge boundary of T, while the other k do contribute. As l is a leaf in T it contributes two edges to T's forward edge boundary. The forward edge boundary of T therefore consists of $k + 1 - 1 + 2 = k + 2$ edges. □

The following corollary of this fact proves that our setup in the previous section of moving from one dynasty to the next is consistent:

Corollary 4.4. *A forest consisting of s disjoint subtrees of the binary tree, $T_1, \dots, T_s$, with $k_1, \dots, k_s$ vertices respectively has a forward edge boundary consisting of*

$$s + \sum_{i=1}^{s} k_i$$

edges. In particular, we have that

$$Z_0^{(n)} = \mathcal{P}^{(n-1)} + Z_0^{(n-1)} \qquad \forall n \in \mathbb{N}.$$

Proof. The first part is trivial given Proposition 4.3. The second part comes from the first, noting that the number of trees in the $(n-1)$th dynasty is $Z_0^{(n-1)}$. □

4.2.3 Progeny of a critical Galton-Watson forest

Consider the Branching Process $(Z_n)_{n \geq 0}$ given by $Z_0 = z_0$ and offspring distribution L where

$$\mathbb{P}(L = 0) = \frac{1}{2}\mathbb{P}(L = 1) = \mathbb{P}(L = 2) = \frac{1}{4}.$$

In other words, the PGF of L is given by $g(s) = \frac{1}{4} + \frac{1}{2}s + \frac{1}{4}s^2$. We can immediately see that $\mathbb{E}[L] = g'(1) = 1$ and $\sigma^2 = Var[L] = g''(1) + g'(1) - (g'(1))^2 = \frac{1}{2}$. So this is a critical branching process with finite variance.

We now define $\mathcal{P}_n = \sum_{i=0}^{n} Z_i$, the progeny up til the nth generation, and $\mathcal{P} = \sum_{i=0}^{\infty} Z_i$, the total progeny. Let $\mathbb{P}_i$ refer to the probability measure on the BP starting with $Z_0 = i$. The following lemma is not a new result, but we are not aware of it being written anywhere in an explicit form, so we shall state and prove it now.

Lemma 4.5. *$\forall z = z(a)$ such that $1 \leq z(a) \leq \sqrt{a}$:*

$$\mathbb{P}_z(\mathcal{P} = a) = (z+1)\pi^{-1/2}a^{-3/2}e^{-\frac{(z+1)^2}{a}}(1 + O(a^{-1}))$$

and for $b \geq a > 1$ ($b = \infty$ is allowed with the obvious interpretation)

$$\mathbb{P}_z(a \leq \mathcal{P} \leq b) \geq \frac{2(z+1)}{\sqrt{a\pi}}\left(1 - \sqrt{\frac{a}{b+1}}\right)\left(1 + O(z^2a^{-1})\right)$$
$$\mathbb{P}_z(a \leq \mathcal{P} \leq b) \leq \frac{2(z+1)}{\sqrt{(a-1)\pi}}\left(1 - \sqrt{\frac{a-1}{b}}\right)\left(1 + O(z^2a^{-1})\right).$$

Proof. Our strategy here is to compare our critical branching process with hitting times of a random walk.

Let $W = (W_n)_{n\geq 0}$ be a random walk on $\mathbb{Z}$ started at z with jump distribution ν satisfying $\mathbb{P}(\nu = m) = \mathbb{P}(L = m+1)$ for all $m \geq -1$. In our specific case W is a lazy simple symmetric random walk started from z where it is lazy (i.e. doesn't move) with probability 1/2. Now [30, Corollary 1.6] tells us that $\mathcal{P} = T$ in distribution, where

$$T = \inf\{n \geq 1 \ : \ W_n = -1\}.$$

Defining $S = (S_i)$ to be the non-negative RW defined by $S_i = W_i + i - z$ we get:

$$\begin{aligned}\mathbb{P}_z(T = a) &= \mathbb{P}_z(W_i \text{ first hits } -1 \text{ on } a\text{th step})\\ &= \mathbb{P}_0(S_i \text{ first hits } i - z - 1 \text{ on } a\text{th step}) = \mathbb{P}_0(T_{-z-1} = a)\end{aligned}$$

where $T_{-z-1} = \inf\{i \geq 1 \ : \ S_i = i - z - 1\}$ and the subscript on $\mathbb{P}$ denotes the starting position of the walk we are considering. By definition we have that S is a random walk started from 0 that makes non-negative steps, so we can use Equation (6.3) in [39] to get that:

$$\begin{aligned}\mathbb{P}_0(T_{-z-1}=a) &= \frac{z+1}{a}\mathbb{P}_0(S_a=a-z-1)\\ &= \frac{z+1}{a}\mathbb{P}_z(W_a=-1)=\frac{z+1}{a}\mathbb{P}_0(W_a=-z-1). \end{aligned} \tag{4.1}$$

We now want to simplify this probability. We can use the LCLT [29, Theorem 2.1.1] to say that if $z \le \sqrt{a}$ then

$$\mathbb{P}_0(W_a=-z-1)=\pi^{-1/2}a^{-1/2}e^{-\frac{(z+1)^2}{a}}(1+O(a^{-1}))$$

so plugging this into (4.1), we have that for $z \le \sqrt{a}$:

$$\mathbb{P}_z(\mathcal{P}=a)=(z+1)\pi^{-1/2}a^{-3/2}e^{-\frac{(z+1)^2}{a}}(1+O(a^{-1})).$$

To conclude the proof we now consider

$$\mathbb{P}_z(a\le\mathcal{P}\le b)=(z+1)\pi^{-1/2}\sum_{k=a}^{b}k^{-3/2}e^{-\frac{(z+1)^2}{k}}(1+O(k^{-1}))$$

where we have $z \le \sqrt{a}$ and $a > 1$. As $k^{-3/2}e^{-1/k}$ is decreasing in k, we know that $\int_a^{b+1} \le \sum_a^b \le \int_{a-1}^b$. Taking the lower bound integral, letting $y = x^{-1/2}$ and using the power series expansion of $\exp(-y^2)$ we obtain

$$\begin{aligned}&\int_a^{b+1} x^{-3/2}e^{-(z+1)^2/x}(1+O(x^{-1}))\ \mathrm{d}x\\ &\qquad= 2\int_{(b+1)^{-1/2}}^{a^{-1/2}} e^{-(z+1)^2y^2}(1+O(y^2))\ \mathrm{d}y\\ &\qquad= 2\int_{(b+1)^{-1/2}}^{a^{-1/2}} (1+O((z+1)^2y^2))(1+O(y^2))\ \mathrm{d}y\\ &\qquad= 2\int_{(b+1)^{-1/2}}^{a^{-1/2}} 1+O((z+1)^2y^2)\ \mathrm{d}y.\end{aligned}$$

Evaluating this integral, the above is

$$\begin{aligned}
&= \frac{2}{\sqrt{a}} - \frac{2}{\sqrt{b+1}} + O\left((z+1)^2(a^{-3/2} - (b+1)^{-3/2})\right) \\
&= \frac{2}{\sqrt{a}}\left(1 - \sqrt{\frac{a}{b+1}}\right)\left(1 + O\left((z+1)^2\frac{a^{-3/2} - (b+1)^{-3/2}}{a^{-1/2} - (b+1)^{-1/2}}\right)\right) \\
&= \frac{2}{\sqrt{a}}\left(1 - \sqrt{\frac{a}{b+1}}\right)\left(1 + O\left(\frac{(z+1)^2}{b+1} + \frac{(z+1)^2}{\sqrt{a(b+1)}} + \frac{(z+1)^2}{a}\right)\right) \\
&= \frac{2}{\sqrt{a}}\left(1 - \sqrt{\frac{a}{b+1}}\right)\left(1 + O\left(z^2 a^{-1}\right)\right)
\end{aligned}$$

where the last line comes from $b \geq a$. The upper bound comes from an almost identical calculation. $\square$

4.2.4 Survival of a critical Galton-Watson forest

In this section we prove the following:

Lemma 4.6. *For a Galton-Watson process with $Z_0 = z$ intial particles and PGF $g(s) = \frac{1}{4} + \frac{1}{2}s + \frac{1}{4}s^2$, we have for $z \leq n$*

$$\mathbb{P}_z(Z_n > 0) \geq \frac{z}{2n}.$$

Before we prove this result, we state our main tool for the proof, [2, Theorem 1]:

Theorem 4.7. *For a branching process (Z_n) with PGF $g(s) = \sum_{j=0}^{\infty} p_j s^j$ such that $g'(1) \leq 1$ and $g''(1) < \infty$. We have*

$$\frac{n}{n+D} \leq \mathbb{P}_1(Z_n = 0) \leq \frac{n}{n + (g''(1))^{-1}}$$

where $D = \max\{2, g'(1)/(p_0 + g'(1) - 1)\}$.

Proof of Lemma 4.6. By Theorem 4.7, a branching process starting with 1 individual with our PGF satisfies

$$\frac{n}{n+4} \leq \mathbb{P}_1(Z_n = 0) \leq \frac{n}{n+2}.$$

Now going back to $\mathbb{P}_z(Z_n > 0)$ we find using independence that

$$\mathbb{P}_z(Z_n > 0) = 1 - \mathbb{P}_z(Z_n = 0) = 1 - (\mathbb{P}_1(Z_n = 0))^z.$$

Thus by Theorem 4.7 we have that

$$1 - \left(\frac{n}{n+2}\right)^z \leq \mathbb{P}_z(Z_n > 0) \leq 1 - \left(\frac{n}{n+4}\right)^z.$$

We focus on the lower bound, which is

$$\begin{aligned} 1 - \left(1 - \frac{2}{n+2}\right)^j &\geq 1 - \left(\exp\left(-\frac{2}{n+2}\right)\right)^j \\ &= 1 - \exp\left(-\frac{2j}{n+2}\right) \\ &\geq 1 - \exp\left(-\frac{j}{n}\right) \\ &\geq \frac{1}{2}\frac{j}{n} \end{aligned}$$

where the penultimate inequality holds provided $n \geq 2$ and and last equality holds provided $j \leq n$ which it does by assumption. For $n = 1$ we must have $z = 1$ and $\mathbb{P}_1(Z_1 > 0) = 3/4 > 1/2$ so it works. $\square$

Note that Theorem 4.7 can be used to produce an upper bound for $\mathbb{P}_z(Z_n > 0)$ but it, in general, is not very accurate and we do not use it in this thesis.

4.3 Outline and ideas behind Conjecture 4.1

4.3.1 Sketch proof of Conjecture 4.1

For Conjecture 4.1, we start by fixing $\gamma > 2$ throughout. We want to prove there exists a time t where we have that

$$\liminf_{n\to\infty} \frac{\log(\gamma) M_n(t)}{\log\log(n)} \leq 1 \ a.s.$$

By the idea of dynasties, we know that if the mth dynasty survives for at least k generations, then the left-most particle after k generations must be $\leq m$, that is, $M_k \leq m$. What we need to show is that for any $\varepsilon > 0$ there exists a time t where there exists an N such that for all $n \geq N$ we have that

$$M_n(t) \leq m_\varepsilon(n) := \frac{\log\log(n)}{\log(\gamma - \varepsilon)}.$$

We often suppress n and simply write m_ε. To this end, for $\varepsilon > 0$ define

$$F_j^\varepsilon(t) := \{Z^{(j)}_{e^{(\gamma-\varepsilon)^j}}(t) > 0\}$$
$$\tilde{F}_k^\varepsilon(t) := \bigcap_{j=0}^{k} F_j^\varepsilon(t)$$

so $F_j^\varepsilon(t)$ is the event at time t the jth dynasty has survived at least $\exp((\gamma-\varepsilon)^j)$ generations. While $\tilde{F}_k^\varepsilon(t)$ requires this behavior from the first $k+1$ dynasties. If we can prove that for any $\varepsilon > 0$ with positive probability there exists an exceptional time $t \in [0,1]$ where $\tilde{F}^\varepsilon_{m_\varepsilon(n)}(t)$ occurs, then at such a t

$$M_{e^{(\gamma-\varepsilon)^j}}(t) \leq j \quad \forall j \leq m_\varepsilon(n) \tag{4.2}$$

Proving that such a t exists with positive probability, so that we can obtain (4.2), is the crux of our entire argument and is where most of our effort must be dedicated.

Moving onward, we can invert the subscript of $M(t)$ in (4.2) with respect to j to obtain that

$$\bigcap_{j=0}^{n} \{M_j(t) \leq m_\varepsilon(j)\}$$

occurs with positive probability. Using this together with some technical arguments, we can show that with positive probability there exists a time $t \in [0,1]$ where

$$\bigcap_{n} \bigcap_{j=0}^{n} \{M_j(t) \leq m_\varepsilon(j)\}.$$

From here, using a 0-1 law, we can prove that there exists N such that

$$\bigcap_{n\geq N} \bigcap_{j=0}^{n} \{M_j(t) \leq m_\varepsilon(j)\}$$

occurs a.s. as required.

We now return to the step leading to (4.2), which is fundamental to our proof, and discuss our strategy for how we hope to prove it. Recall that we need to prove that with positive probability times exist where $\tilde{F}^\varepsilon_{m_\varepsilon(n)}(t)$ occurs. To this end we define for any collection of events $E = (E(t))_{t\in[0,1]}$ that are defined on our model

$$\kappa(E) = \int_0^1 \mathbb{1}_{E(t)} \, \mathrm{d}t$$

the Lebesgue amount of time in $[0,1]$ where E occurs. We would like to use the second moment method on this random variable for our event $\tilde{F}^{\varepsilon}_{m_\varepsilon(n)}$, i.e. show that $\mathbb{E}[\kappa(\tilde{F}^{\varepsilon}_{m_\varepsilon(n)})^2] \le C\mathbb{E}[\kappa(\tilde{F}^{\varepsilon}_{m_\varepsilon(n)})]^2$ for some constant $C > 0$, as standard literature then implies exceptional times exist (given some technical results which we prove). A key insight is that it turns out to be much easier to work with events focusing on progeny rather than survival, and so our strategy will be to show that exceptional times exist for the event that for the first $j \le m_\varepsilon(n)$ dynasties, each has a tree that has progeny approximately e^{γ^j}. We then prove that exceptional times for this event imply exceptional times for this event combined with $\tilde{F}^{\varepsilon}_{m_\varepsilon(n)}(t)$, immediately implying that there are exceptional times where $\tilde{F}^{\varepsilon}_{m_\varepsilon(n)}(t)$ occurs.

4.3.2 Proof outline of Conjecture 4.1

We now give a more detailed outline of the proof, breaking it down into smaller results which we prove in later sections.

Let $\gamma > 2$ be fixed, and $(\varepsilon_j)_{j\ge 0}$ be a sequence of numbers in $[0,1]$ satisfying $\varepsilon_j = j^{-2}$ for $j \ge 1$, and $3/e < 1 + \varepsilon_0 < 4/e$. Define for $t \ge 0$

$$
\begin{aligned}
A'_j(t) &:= \{\exists \text{ tree in } j\text{th dynasty at time } t \text{ with progeny} \in [e^{\gamma^j}, (1+\varepsilon_j)e^{\gamma^j}]\} \\
A_j(t) &:= A'_j(t) \cap \{\mathcal{P}^{(j)}(t) \le (1+2\varepsilon_j)e^{\gamma^j}\}.
\end{aligned}
$$

We also define the event that the first $k+1$ dynasties have suitably large progenies:

$$B_k(t) := \bigcap_{j=0}^{k} A_j(t).$$

Finally we define, for any $0 < \varepsilon < \gamma - 2$, the event that the first $k+1$ dynasties live sufficiently long and have large progeny:

$$\tilde{B}^{\varepsilon}_k(t) := \bigcap_{j=0}^{k} \big(A_j(t) \cap F^{\varepsilon}_j(t)\big) = B_k(t) \cap \tilde{F}^{\varepsilon}_k(t)$$

where F^{ε}_j and $\tilde{F}^{\varepsilon}_k$ are defined as in Section 4.3.1. For the collections of events we have just defined, we will be interested in whether they occur at two different times simultaneously. To this end we shall use the shorthand

$$A_j(s,t) := A_j(s) \cap A_j(t)$$

etc. when appropriate. Recalling the definition of κ from Section 4.3.1, the key result we do not have the proof for is that the second moment method holds for $(B_{m_\varepsilon(n)})_n$:

Conjecture 4.8. *For all $\varepsilon > 0$, there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$*

$$\mathbb{E}[\kappa(B_{m_\varepsilon(n)})^2] \leq C\mathbb{E}[\kappa(B_{m_\varepsilon(n)})]^2.$$

We prove in Section 4.6.1 that the above is sufficient for proving a second moment bound for $(\tilde{B}_{m_\varepsilon(n)})_n$, the collection of events that link progeny to survival:

Lemma 4.9. *If $\mathbb{E}[\kappa(B_{m_\varepsilon(n)})^2] \leq C_1\mathbb{E}[\kappa(B_{m_\varepsilon(n)})]^2$ for some constant $C_1 > 0$, then there exists a constant $C_2 > 0$ such that*

$$\mathbb{E}[\kappa(\tilde{B}^\varepsilon_{m_\varepsilon(n)})^2] \leq C_2\mathbb{E}[\kappa(\tilde{B}^\varepsilon_{m_\varepsilon(n)})]^2.$$

By Lemma 4.9 and the Cauchy-Schwarz inequality, we have that

$$\mathbb{P}(\kappa(\tilde{B}^\varepsilon_{m_\varepsilon(n)}) > 0) \geq \frac{\mathbb{E}[\kappa(\tilde{B}^\varepsilon_{m_\varepsilon(n)})]^2}{\mathbb{E}[\kappa(\tilde{B}^\varepsilon_{m_\varepsilon(n)})^2]} \geq C_2^{-1} > 0.$$

We can then use countable additivity (as $(\tilde{B}^\varepsilon_{m_\varepsilon(n)})_n$ is decreasing), and the fact that $\tilde{B}^\varepsilon_{m_\varepsilon(n)}(t) \subset \tilde{F}_{m_\varepsilon(n)}(t)$ to obtain

$$\mathbb{P}\left(\bigcap_{n\geq 1}\{(\kappa(\tilde{F}^\varepsilon_{m_\varepsilon(n)}) > 0\}\right) \geq \mathbb{P}\left(\bigcap_{n\geq 1}\{(\kappa(\tilde{B}^\varepsilon_{m_\varepsilon(n)}) > 0\}\right) \geq C_2^{-1} > 0. \tag{4.3}$$

We now have the following Lemma, which will be proved in Section 4.6.2:

Lemma 4.10. *Define $T_n := \{t \in [0,1] \ : \ M_j(t) \leq m_\varepsilon(j) \ \forall j \leq n\}$. Then*

$$\bigcap_{n=1}^{\infty} \bar{T}_n = \bigcap_{n=1}^{\infty} T_n \ a.s.$$

The utility of this theorem is as follows, T_n is the set of times where $\tilde{F}^\varepsilon_{m_\varepsilon(n)}(t)$ occurs, ergo

$$\kappa(\tilde{F}^\varepsilon_{m_\varepsilon(n)}) > 0 \iff T_n \neq \emptyset$$

so that (4.3) reads

$$\mathbb{P}\left(\bigcap_{n\geq 1}\{T_n \neq \emptyset\}\right) \geq C_2^{-1} > 0.$$

Now via compactness and Lemma 4.10

$$\left\{\bigcap_n T_n \neq \emptyset\right\} = \left\{\bigcap_n \bar{T}_n \neq \emptyset\right\} = \bigcap_n \{\bar{T}_n \neq \emptyset\} \supseteq \bigcap_n \{T_n \neq \emptyset\}$$

and it is clear that

$$\bigcap_{n=1}^{\infty} T_n = \{t \in [0,1] \ : \ M_j(t) \leq m_\varepsilon(j) \ \forall j\} = \{t \in [0,1] \ : \ \bigcap_n \tilde{F}^\varepsilon_{m_\varepsilon(n)}(t) \text{ occurs}\}$$

so

$$\begin{aligned}
\mathbb{P}(\exists t \in [0,1] \ : \ \cap_n \tilde{F}^\varepsilon_{m_\varepsilon(n)}(t) \text{ occurs}) &= \mathbb{P}(\cap_n T_n \neq \emptyset) \\
&\geq \mathbb{P}(\cap_n \{T_n \neq \emptyset\}) \\
&= \mathbb{P}\left(\bigcap_{n\geq 1} \{(\kappa(\tilde{F}^\varepsilon_{m_\varepsilon(n)}) > 0\}\right) \\
&\geq C_2^{-1} > 0.
\end{aligned}$$

This means, for any $\varepsilon > 0$, that with positive probability we have an exceptional time where $M_n(t) \leq m_\varepsilon(n) \ \forall n$. If we now define

$$H_0^\varepsilon(t) := \left\{\forall \text{large } n \ : \ M_n(t) \leq \frac{\log\log(n)}{\log(\gamma - \varepsilon)}\right\}$$

which is the event we want to show occurs a.s. for some t (for every $\varepsilon > 0$). Then we have that

$$\mathbb{P}(\exists t \in [0,1] \ : \ H_0^\varepsilon(t) \text{ occurs}) \geq \mathbb{P}\left(\bigcap_{n\geq 1} \{(\kappa(\tilde{F}^\varepsilon_{m_\varepsilon(n)}) > 0\}\right) \geq C^{-1} > 0.$$

We finally prove the following in Section 4.6.2:

Lemma 4.11. $\{\exists t \in [0,1] \ : \ H_0^\varepsilon(t)$ *occurs*$\}$ *satisfies a 0-1 law. That is, it either occurs a.s. or does not occur a.s.*

which, combined with the fact we know this event occurs with positive probability, concludes the proof.

4.4 Lower bound on the first moment

We produce a lower bound for the squared first moment, $\mathbb{E}[\kappa(B_{m_\varepsilon(n)})]^2$. Using Fubini's theorem, stationarity, and telescoping we obtain (abbreviating $m_\varepsilon(n)$ as m_ε)

$$\mathbb{E}[\kappa(B_{m_\varepsilon})] = \int_0^1 \mathbb{P}(B_{m_\varepsilon}(t))\,\mathrm{d}t = \mathbb{P}(B_{m_\varepsilon}(0))$$
$$= \mathbb{P}(A_0(0)) \prod_{j=0}^{m_\varepsilon - 1} \mathbb{P}\left(A_{j+1}(0) \,\middle|\, \bigcap_{i=0}^{j} A_i(0)\right). \tag{4.4}$$

As we defined ε_0 to satisfy $3/e < 1 + \varepsilon_0 < 4/e$, we have

$$\mathbb{P}(A_0(0)) = \mathbb{P}_1(\mathrm{prog} \in [e, (1+\varepsilon_0)e]) = \mathbb{P}_1(\mathrm{prog} = 3) = 5/64.$$

To compute the product, we have the following lemma which shall be proved shortly:

Lemma 4.12. *For $j \geq 0$ we have*

$$\mathbb{P}\left(A_{j+1}(0) \,\middle|\, \bigcap_{i=0}^{j} A_i(0)\right) \geq \frac{2\varepsilon_{j+1}}{\sqrt{\pi}} e^{\gamma^j - \frac{1}{2}\gamma^{j+1}} (1 + O(\varepsilon_{j+1})).$$

As $\varepsilon_{j+1} = (j+1)^{-2}$, elementary algebra gives that

$$\prod_{j=0}^{\infty} (1 + O(\varepsilon_{j+1}))^2 = C < \infty.$$

Now substituting Lemma 4.12 into (4.4) gives

$$\mathbb{P}(B_{m_\varepsilon}(0))^2 \geq C \left(\frac{5}{64}\right)^2 4^{m_\varepsilon} \pi^{-m_\varepsilon} \left(\prod_{j=0}^{m_\varepsilon - 1} \varepsilon_{j+1}^2\right) \exp\left(\sum_{j=0}^{m_\varepsilon - 1} \left(2\gamma^j - \gamma^{j+1}\right)\right)$$
$$= C' 4^{m_\varepsilon} \pi^{-m_\varepsilon} \left(\prod_{j=0}^{m_\varepsilon - 1} \varepsilon_{j+1}^2\right) \exp\left(-\left(\frac{\gamma - 2}{\gamma - 1}\right)(\gamma^{m_\varepsilon} - 1)\right). \tag{4.5}$$

This is our lower bound on the first moment, we hope to get a second moment bound to the same level of accuracy. We dedicate the rest of this section to proving Lemma 4.12.

Proof of Lemma 4.12. Due to the $O(\varepsilon_{j+1})$ term in the statement of our result, we need only focus on large j in this proof. So we view j as large throughout.

Given $A_j(0)$, we know that $\mathcal{P}^{(j)}(0) \in [e^{\gamma^j}, (1+2\varepsilon_j)e^{\gamma^j}]$ and so the number of trees in the $(j+1)$th forest is at least $e^{\gamma^j}+1$, and at most $2(1+2\varepsilon_j)e^{\gamma^j}$. The minimum is the case where $\mathcal{P}^{(j)}(0) = e^{\gamma^j}$ and consists of only one tree, while the maximum is where $\mathcal{P}^{(j)}(0) = (1+2\varepsilon_j)e^{\gamma^j}$ and there are precisely that many trees (and each have 2 edges on their forward edge boundary). We have

$$\begin{aligned}\mathbb{P}\left(A_{j+1}(0) \,\middle|\, \bigcap_{i=0}^{j} A_i(0)\right) &\geq \min_{z\in[e^{\gamma^j}, 2(1+2\varepsilon_j)e^{\gamma^j}]} \mathbb{P}_z(A_{j+1}(0)) \\ &\geq \min_{z\in[e^{\gamma^j}, 2(1+2\varepsilon_j)e^{\gamma^j}]} \mathbb{P}_z(A''_{j+1}(0))\end{aligned}$$

where $A''_{j+1}(0)$ is the event that exactly one tree in the forest in position $j+1$ at time 0 has progeny in $[e^{\gamma^{j+1}}, (1+\varepsilon_{j+1})e^{\gamma^{j+1}}]$ and the total progeny of all the other trees in said forest is at most $\varepsilon_{j+1}e^{\gamma^{j+1}}$. Clearly $A''_{j+1}(0) \subset A_{j+1}(0)$. We have that

$$\mathbb{P}_z(A''_{j+1}(0)) = z\mathbb{P}_1(\mathcal{P}^{(j+1)}(0) \in [e^{\gamma^{j+1}}, (1+\varepsilon_{j+1})e^{\gamma^{j+1}}])\mathbb{P}_{z-1}(\mathcal{P}^{(j+1)}(0) \leq \varepsilon_{j+1}e^{\gamma^{j+1}})$$

where we've used that there are z candidates for which tree is the large one, and the fact that said tree is independent of all the other trees. We can use Lemma 4.5 on the first probability to obtain that

$$\begin{aligned}\mathbb{P}_1(\mathcal{P}^{(j+1)}(0) &\in [e^{\gamma^{j+1}}, (1+\varepsilon_{j+1})e^{\gamma^{j+1}}]) \\ &\geq \frac{4e^{-\frac{1}{2}\gamma^{j+1}}}{\sqrt{\pi}}\left(1-\sqrt{\frac{1}{1+\varepsilon_{j+1}+e^{-\gamma^{j+1}}}}\right)\left(1+O(e^{-\gamma^{j+1}})\right) \\ &\geq \frac{2\varepsilon_{j+1}e^{-\frac{1}{2}\gamma^{j+1}}}{\sqrt{\pi}}\left(1+O(\varepsilon_{j+1})\right)\left(1+O(e^{-\gamma^{j+1}})\right) \\ &= \frac{2\varepsilon_{j+1}e^{-\frac{1}{2}\gamma^{j+1}}}{\sqrt{\pi}}\left(1+O(\varepsilon_{j+1})\right)\end{aligned}$$

since $1-(1+x)^{-1/2} = x/2+O(x^2)$ as $x \to 0$. The last line holds since $\varepsilon_j = j^{-2}$ which decays slower than $\exp(-\gamma^{j+1})$. For the second probability, we use Lemma 4.5 on its complement, which we are allowed to do as $z-1$ is at most $2(1+2\varepsilon_j)e^{\gamma^j}-1$ which is smaller than $\sqrt{\varepsilon_{j+1}e^{\gamma^{j+1}}}$ for j that is not too small. To summarise, we have

$$\begin{aligned}\mathbb{P}_{z-1}(\mathcal{P}^{(j+1)}(0) \leq \varepsilon_{j+1}e^{\gamma^{j+1}}) &= 1-\mathbb{P}_{z-1}(\mathcal{P}^{(j+1)}(0) > \varepsilon_{j+1}e^{\gamma^{j+1}}) \\ &\leq 1-\frac{2z}{\sqrt{\pi(\varepsilon_{j+1}e^{\gamma^{j+1}}-1)}}\left(1+O\left(z^2\varepsilon_{j+1}^{-1}e^{-\gamma^{j+1}}\right)\right) \to 1\end{aligned}$$

as $j \to \infty$. The last line above, pre-limit, is an increasing function of j. In particular, the above expression is at least a constant $C > 0$ for all j, regardless of z (within the region z is allowed to be in). Combining all of this, and absorbing C into the O, we have

$$\mathbb{P}_z(A''_{j+1}(0)) \geq \frac{2\varepsilon_{j+1} z}{\sqrt{\pi}} e^{-\frac{1}{2}\gamma^{j+1}}(1 + O(\varepsilon_{j+1})). \tag{4.6}$$

Thus our lower bound is

$$\begin{aligned}
\mathbb{P}\left(A_{j+1}(0) \,\middle|\, \bigcap_{i=0}^{j} A_i(0)\right) &\geq \min_{z \in [e^{\gamma^j}, 2(1+2\varepsilon_j)e^{\gamma^j}]} \mathbb{P}_z(A''_{j+1}(0)) \\
&\geq \min_{z \in [e^{\gamma^j}, 2(1+2\varepsilon_j)e^{\gamma^j}]} \frac{2\varepsilon_{j+1} z}{\sqrt{\pi}} e^{-\frac{1}{2}\gamma^{j+1}}(1 + O(\varepsilon_{j+1})) \\
&= \frac{2\varepsilon_{j+1}}{\sqrt{\pi}} e^{\gamma^j - \frac{1}{2}\gamma^{j+1}}(1 + O(\varepsilon_{j+1}))
\end{aligned}$$

as required. □

4.5 Progress on Conjecture 4.8: the 2nd moment

We start by detailing the second moment computation, highlighting what we need to bound, and give heuristics for what we expect to happen. In the second subsection we prove results we expect to aid us in computing the second moment bounds.

4.5.1 The second moment computation

For the 2nd moment, Fubini's theorem and stationarity gives

$$\begin{aligned}
\mathbb{E}[\kappa(B_{m_\varepsilon})^2] &= \mathbb{E}\left[\left(\int_0^1 \mathbb{1}_{B_{m_\varepsilon}(t)}\,\mathrm{d}t\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_0^1 \mathbb{1}_{B_{m_\varepsilon}(t)}\,\mathrm{d}t\right)\left(\int_0^1 \mathbb{1}_{B_{m_\varepsilon}(s)}\,\mathrm{d}s\right)\right] \\
&= \mathbb{E}\left[\int_0^1\int_0^1 \mathbb{1}_{B_{m_\varepsilon}(s,t)}\,\mathrm{d}s\,\mathrm{d}t\right] \leq 2\int_0^1 \mathbb{P}(B_{m_\varepsilon}(0,t))\,\mathrm{d}t.
\end{aligned}$$

We must bound $2\int_0^1 \mathbb{P}(B_{m_\varepsilon}(0,t))\,\mathrm{d}t$ from above. With telescoping we get:

$$\begin{aligned}\mathbb{P}(B_{m_\varepsilon}(0,t)) &= \mathbb{P}(A_0(0,t)) \prod_{j=0}^{m_\varepsilon-1} \mathbb{P}\left(A_{j+1}(0,t) \;\middle|\; \bigcap_{i=0}^{j} A_i(0,t)\right) \\ &\leq \prod_{j=0}^{m_\varepsilon-1} \mathbb{P}\left(A_{j+1}(0,t) \;\middle|\; \bigcap_{i=0}^{j} A_i(0,t)\right).\end{aligned}$$

This means that the quantity of interest is

$$2\int_0^1 \prod_{j=0}^{m_\varepsilon-1} \mathbb{P}\left(A_{j+1}(0,t) \;\middle|\; \bigcap_{i=0}^{j} A_i(0,t)\right)\,\mathrm{d}t.$$

What we require is an extremely accurate upper bound on $\mathbb{P}(A_{j+1}(0,t) \mid \cap_i A_i(0,t))$ so that we can bound the entire product, hence the integral, precisely. A bound to this level of precision is what we are currently lacking. However, we have already made some progress in analysing this probability, which we now detail.

Lemma 4.13. *For $j \geq 0$*

$$\begin{aligned}\mathbb{P}\left(A_{j+1}(0,t) \;\middle|\; \bigcap_{i=0}^{j} A_i(0,t)\right) \leq &\max_{z\in[0,2(1+2\epsilon_j)e^{\gamma^j}]} (z\mathbb{P}_1(A'_{j+1}(0,t)) \wedge \mathbb{P}_z(A'_{j+1}(0))) \\ &+ \max_{z\in[e^{\gamma^j},2(1+2\epsilon_j)e^{\gamma^j}]} z^2\mathbb{P}_1(A'_{j+1}(0))^2.\end{aligned}$$

Proof. We start off by defining

$$E_j(t) := \{\mathcal{P}^{(j)}(t) \in [e^{\gamma^j}, (1+2\varepsilon_j)e^{\gamma^j}]\}.$$

We now have the inequality

$$\begin{aligned}\mathbb{P}\left(A_{j+1}(0,t) \;\middle|\; \bigcap_{i=0}^{j} A_i(0,t)\right) &\leq \mathbb{P}\left(A_{j+1}(0,t) \;\middle|\; \bigcap_{i=0}^{j} E_i(0,t)\right) \\ &\leq \mathbb{P}\left(A'_{j+1}(0,t) \;\middle|\; \bigcap_{i=0}^{j} E_i(0,t)\right).\end{aligned}$$

The justification for the first inequality is as follows. While both $\cap_i A_i(0,t)$ and $\cap_i E_i(0,t)$ provide the same information regarding the total progeny of each dynasty, the former concentrates most of that progeny on one specific tree per dynasty, while for $\cap_i E_i(0,t)$ the offspring can be spread over more trees. By Corollary 4.4, we therefore see that conditioning on $\cap_i E_i(0,t)$ rather than $\cap_i A_i(0,t)$ increases the number of initial particles in the $(j+1)$th dynasty at both times, which in turn increases the chance of $A_{j+1}(0,t)$ occurring.

There are two ways that $\mathbb{P}(A'_{j+1}(0,t) \mid \cap_i E_i(0,t))$ can occur. The first way is that a single edge in the forward edge boundary of both the time 0 and time t jth dynasties gives rise to the large progeny required from the $(j+1)$th dynasty at both times. Given $\cap_i E_i(0,t)$ this occurs with probability

$$\leq \max_{z \in [0, 2(1+2\epsilon_j)e^{\gamma^j}]} z\mathbb{P}_1(A'_{j+1}(0,t))$$

as the total progeny in the jth dynasty at both times is at most $(1+2\epsilon_j)e^{\gamma^j}$, each of which contributes at most two to the forward edge boundary. We can then take the minimum of this probability with $\mathbb{P}_z(A'_{j+1}(0))$ since it definitely must occur at time 0 for it to work at both times. The min is needed in the first place to stop $z\mathbb{P}_1(A'_{j+1}(0,t))$ exceeding one when z is too large or t is too small.

The other way $\mathbb{P}(A'_{j+1}(0,t) \mid \cap_i E_i(0,t))$ may occur is if two different edges, one in the forward edge boundary of the time 0 jth dynasty, the other in the forward edge boundary of the time t jth dynasty, each produces a large tree in the $(j+1)$th dynasty at their respective times. Given the information regarding the jth dynasty, from the event $E_j(0,t)$, this occurs with probability

$$\leq \max_{z \in [e^{\gamma^j}, (1+2\epsilon_j)e^{\gamma^j}]} z\mathbb{P}_1(A'_{j+1}(0)) \max_{y \in [e^{\gamma^j}, (1+2\epsilon_j)e^{\gamma^j}]} y\mathbb{P}_1(A'_{j+1}(t))$$

which of course, leads to our required bound via time invariance. □

Lemma 4.13 produces two probabilities that we need to study, $\mathbb{P}_1(A'_{j+1}(0,t))$ and $\mathbb{P}_z(A'_{j+1}(0))$ (for a range of z). The latter does not involve dynamical time t, so should be possible to bound similarly to the first moment. The former however is quite challenging, it appears that for fixed j, the dominant factors affecting $\mathbb{P}(A_{j+1}(0,t) \mid A_j(0,t))$ depend on t in that there are three regimes. One where $t < \exp(-\gamma^{j+1})$, one where $t > \exp(-\gamma^j)$, and one where t is between $\exp(-\gamma^{j+1})$ and $\exp(-\gamma^j)$. To try and capture this we aim to split the integral over $[0,1]$ into a sum of integrals, each over the intervals $[0, e^{-\gamma^{m_\varepsilon}}], [e^{-\gamma^{m_\varepsilon}},\ e^{-\gamma^{m_\varepsilon - 1}}], \ldots, [e^{-1}, 1]$. We can then bound each one separately to get an overall upper bound.

4.5.2 The 0th dynasty at times 0 and t

Recall that the 0th dynasty consists of a single tree containing the root and all vertices connected to said root by solely zero-valued edges. We can view this as a percolation model in that the 0-component containing the root is the 0th dynasty. As our branching random walk evolves over time, this dynasty evolves as well.

Fix $t > 0$, denote by T_0 and T_t the 0-component containing the root at times 0 and t respectively. Define $\tilde{T}_t := T_0 \cap T_t$, the 0-component containing the root at both times 0 and t. This object will be important and thus we shall study its distribution in detail here. For practical purposes we can actually use this object starting not from the root, but with respect to the subtree from some edge we know is one-valued at both times 0 and t.

If we denote the generation sizes of T_0 by $(Z_n)_{n\in\mathbb{N}_0}$, then these form a critical Galton-Watson branching process with offspring distribution L with generating function

$$g(s) = \frac{1}{4} + \frac{1}{2}s + \frac{1}{4}s^2.$$

We prove the following:

Proposition 4.14. *The generation sizes of $\tilde{T}_t$, $(\tilde{Z}_n)_{n\in\mathbb{N}_0}$, follows a Galton-Watson branching process with $\tilde{Z}_0 = 1$ and offspring distribution $\tilde{L}$ with generating function*

$$\tilde{g}(s) = \frac{1}{16}(3-e^{-t})^2 + \frac{s}{8}(1+e^{-t})(3-e^{-t}) + \frac{s^2}{16}(1+e^{-t})^2.$$

In particular $\mathbb{E}[\tilde{L}] = \frac{1}{2}(1+e^{-t}) < 1$ so the process is subcritical. We also have $Var[\tilde{L}] = \frac{1}{8}(1+e^{-t})(3-e^{-t})$.

Proof. Recall that the coefficient of s^i corresponds to $\mathbb{P}(\tilde{L} = i)$. For $\{\tilde{L} = 2\}$ to occur we require that both child edges are zero-valued at times 0 and t. This means that they are both zero-valued at time 0 (occurs with probability 1/4) and between 0 and t each edge independently changes an even number of times, i.e.

$$\mathbb{P}(\tilde{L} = 2) = \frac{1}{4}\mathbb{P}(2 \text{ indep } Po(t/2)'s \in 2\mathbb{Z}) = \frac{1}{16}(1+e^{-t})^2$$

where $2\mathbb{Z}$ denotes the set of even integers, and the probability that a Poisson random variable is even is from [27, Exercise 1.7].

For $\{\tilde{L} = 0\}$ to occur we need that either both edges equal 1 at time 0, both are equal to 0 at time 0 then equal 1 at t, or exactly one edge is equal to 0 at time 0 and that edge is then 1 at t. To go from 0 at time 0 to 1 at time t, or vice-versa, we need an odd number of changes to the status of that edge over $[0, t]$. Utilising [27, Exercise 1.7] again, we compute the following

$$\begin{aligned}\mathbb{P}(\tilde{L}=0) &= \frac{1}{4}+\frac{1}{4}\mathbb{P}(2 \text{ indep } Po(t/2)'s \text{ are off}) + \frac{1}{2}\mathbb{P}(Po(t/2) \notin 2\mathbb{Z}) \\ &= \frac{1}{4}\left(1+\frac{1}{4}(1-e^{-t})^2+(1-e^{-t})\right) = \frac{1}{16}(3-e^{-t})^2.\end{aligned}$$

For $\{\tilde{L}=1\}$ to occur we need both edges to equal 0 at time 0 and exactly one of them to equal 1 at time t (which can be done in two ways), or we need exactly one edge to equal 0 at time 0 and for that edge to stay equal to 0 at time t. This is

$$\begin{aligned}\mathbb{P}(\tilde{L}=1) &= 2\times\frac{1}{4}\mathbb{P}(Po(t/2)\notin 2\mathbb{Z})\mathbb{P}(Po(t/2)\in 2\mathbb{Z}) + \frac{1}{2}\mathbb{P}(Po(t/2)\notin 2\mathbb{Z}) \\ &= \frac{1}{8}(1-e^{-t})(1+e^{-t})+\frac{1}{4}(1+e^{-t}) \\ &= \frac{1}{8}(1+e^{-t})(3-e^{-t})\end{aligned}$$

which gives us the generating function. Finally

$$\mathbb{E}[\tilde{L}] = \tilde{g}'(1) = \frac{1}{2}(1+e^{-t})$$

and

$$\begin{aligned}Var[\tilde{L}] = \tilde{g}''(1)+\tilde{g}'(1)-\tilde{g}'(1)^2 &= \frac{1}{8}(1+e^{-t})^2+\frac{1}{2}(1+e^{-t})-\frac{1}{4}(1+e^{-t})^2 \\ &= \frac{1}{8}(1+e^{-t})(3-e^{-t}).\end{aligned}$$

□

Proposition 4.15.

$$\mathbb{P}(|\tilde{T}_t|=k) = \pi^{-1/2}k^{-3/2}e^{-\frac{1}{k}}\frac{(3-e^{-t})^{k+1}}{4^k(1+e^{-t})^{k-1}}(1+O(k^{-1})).$$

Proof. We start off almost identically to the proof of Lemma 4.5. Define the random walk $W=(W_n)_{n\in\mathbb{N}_0}$ started from 0 with step distribution given by $\tilde{Y}:=\tilde{L}-1$. Also define $S_n = W_n + n$ for all n. By [30, Corollary 1.6] and Equation (6.3) in [39] we have

$$\begin{aligned}\mathbb{P}(|\tilde{T}_t| = k) &= \mathbb{P}(W \text{ first hits } -1 \text{ at step } k)\\ &= \mathbb{P}(S_i = i-1 \text{ for the first time when } i = k)\\ &= \frac{1}{k}\mathbb{P}(S_k = k-1) = \frac{1}{k}\mathbb{P}(W_k = -1). \end{aligned} \tag{4.7}$$

We now use a technique called tilting to change our probability measure from $\mathbb{P}$ to $\mathbb{Q}$ so that under $\mathbb{Q}$, W is a (lazy) simple symmetric random walk. We define this change of measure via the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_j} = \frac{e^{\lambda W_j}}{\mathbb{E}_{\mathbb{P}}[e^{\lambda W_j}]}$$

where $\mathbb{E}_{\mathbb{P}}$ is expectation under $\mathbb{P}$, $\mathcal{F}_j$ is the filtration generated by (W_n), and $\lambda \in \mathbb{R}$ will be chosen such that $\mathbb{E}_{\mathbb{Q}}[W_1] = 0$, making our walk under $\mathbb{Q}$ symmetric. We find this λ now:

$$\mathbb{E}_{\mathbb{Q}}[W_1] = \frac{\mathbb{E}_{\mathbb{P}}[W_1 e^{\lambda W_1}]}{\mathbb{E}_{\mathbb{P}}[e^{\lambda W_1}]}$$

so we need $\mathbb{E}_{\mathbb{P}}[W_1 e^{\lambda W_1}] = 0$. W_1 has the same law as $\tilde{Y} = \tilde{L} - 1$ so

$$\begin{aligned}0 = \mathbb{E}_{\mathbb{P}}[W_1 e^{\lambda W_1}] &= e^{\lambda}\mathbb{P}(\tilde{L} = 2) - e^{-\lambda}\mathbb{P}(\tilde{L} = 0)\\ &= \frac{1}{16}\left(e^{\lambda}(1+e^{-t})^2 + e^{-\lambda}(3-e^{-t})^2\right)\end{aligned}$$

and rearranging gives

$$\lambda = \log\left(\frac{3-e^{-t}}{1+e^{-t}}\right) = t + O(t^2).$$

Now with this value of λ, we use our new measure to compute

$$\begin{aligned}\mathbb{P}(W_k = -1) &= \mathbb{Q}(e^{-\lambda W_k}\mathbb{E}_{\mathbb{P}}[e^{\lambda W_k}]\mathbb{1}_{\{W_k=-1\}})\\ &= e^{\lambda}\mathbb{E}_{\mathbb{P}}[e^{\lambda W_k}]\mathbb{Q}(W_k = -1)\\ &= e^{\lambda}\mathbb{E}_{\mathbb{P}}[e^{\lambda W_1}]^k\mathbb{Q}(W_k = -1)\end{aligned} \tag{4.8}$$

where the last equality comes from W_k having the same law as the sum of k variables each with the same law as W_1. Under $\mathbb{Q}$, W_k is the kth step of a (lazy) simple symmetric random walk so by the LCLT [29, Theorem 2.1.1]

$$\mathbb{Q}(W_k = -1) = \pi^{-1/2}k^{-\frac{1}{2}}e^{-\frac{1}{k}}(1 + O(k^{-1}))$$

We also compute (using the exact value of λ above)

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[e^{\lambda W_1}] &= e^{-\lambda}\mathbb{P}(\tilde{L}=0)+\mathbb{P}(\tilde{L}=1)+e^{\lambda}\mathbb{P}(\tilde{L}=2)\\ &= \frac{1}{16}\left(e^{-\lambda}(3-e^{-t})^2+2(1+e^{-t})(3-e^{-t})+e^{\lambda}(1+e^{-t})^2\right)\\ &= \frac{1}{16}\left((1+e^{-t})(3-e^{-t})+2(1+e^{-t})(3-e^{-t})+(1+e^{-t})(3-e^{-t})\right)\\ &= \frac{1}{4}(1+e^{-t})(3-e^{-t})\end{aligned}$$

Inserting these values into (4.8) and then (4.7) gives

$$\mathbb{P}(|\tilde{T}_t|=k)=k^{-1}\mathbb{P}(W_k=-1)=\pi^{-1/2}k^{-3/2}e^{-\frac{1}{k}}\frac{(3-e^{-t})^{k+1}}{4^k(1+e^{-t})^{k-1}}(1+O(k^{-1}))$$

as required. □

The following lemma will be useful when studying the forward edge boundary of $\tilde{T}_t$. In particular whether edges on said boundary are zero at either times 0 or t, but they cannot be zero at both, because an edge on the boundary of $\tilde{T}_t$ that is on at both times would just be part of $\tilde{T}_t$, not on its boundary.

Lemma 4.16. *Writing $X(t)=X_{i_1\dots i_n}(t)$ for the status of a fixed edge variable $X_{i_1\dots i_n}$ at time t in a dynamical Galton-Watson forest, we have*

$$\mathbb{P}(X(0)=0,\ X(t)=1\mid\{X(0)=1\}\cup\{X(t)=1\})=\frac{1-e^{-t}}{3-e^{-t}}=:p_t.$$

Proof. We have

$$\mathbb{P}(X(0)=0,\ X(t)=1\mid\{X(0)=1\}\cup\{X(t)=1\})=\frac{\mathbb{P}(X(0)=0,\ X(t)=1)}{\mathbb{P}(\{X(0)=1\}\cup\{X(t)=1\})}.$$

We have by [27, Exercise 1.7]

$$\mathbb{P}(X(0)=0,\ X(t)=1)=\frac{1}{2}\mathbb{P}\left(Po(t/2)\notin 2\mathbb{Z}\right)=\frac{1}{4}(1-e^{-t}),$$

while

$$\mathbb{P}(X(0)=X(1)=0)=\frac{1}{2}\mathbb{P}\left(Po(t/2)\in 2\mathbb{Z}\right)=\frac{1}{4}(1+e^{-t}).$$

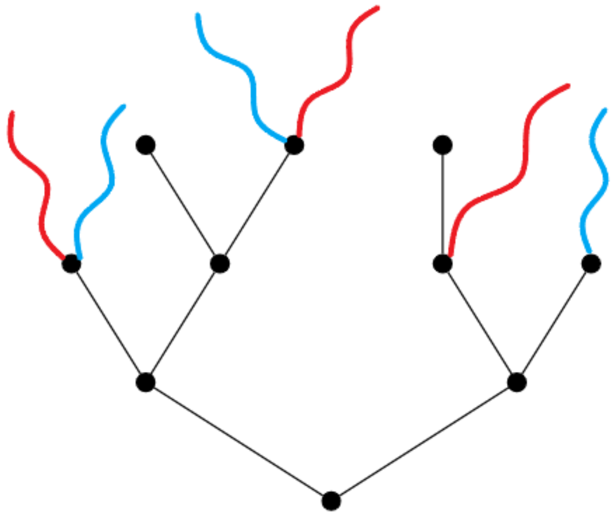

Figure 4-3: The tree $\tilde{T}_t$ (black) with the branches in blue and red representing components in T_0 and T_t respectively.

Therefore

$$\begin{aligned}\mathbb{P}(X(0) = 0,\ X(t) = 1 \mid \{X(0) = 1\} \cup \{X(t) = 1\}) &= \frac{\frac{1}{4}(1 - e^{-t})}{1 - \frac{1}{4}(1 + e^{-t})} \\ &= \frac{1 - e^{-t}}{3 - e^{-t}} =: p_t.\end{aligned}$$

which can be seen to be of order t. □

Let B_0 and B_t be the (random) number of edges on the forward edge boundary of $\tilde{T}_t$ that are on (zero-valued) at times 0 and t respectively. Given $\{|\tilde{T}_t| = z\}$, the number of edges on the forward edge boundary is $z + 1$, and the joint distribution of $(B_0, B_t, z + 1 - B_0 - B_t)$ follows a multinomial distribution with parameters $(z + 1, p_t, p_t, 1 - 2p_t)$, where p_t is the probability from Lemma 4.16. See Figure 4-3. We have the following result:

Lemma 4.17.

$$\mathbb{E}_1[B_0 B_t \mathbb{1}_{\{|\tilde{T}_t| = z\}}] = z(z + 1)p_t^2 \mathbb{P}_1(|\tilde{T}_t| = z).$$

Proof. We start off by using the tower property to introduce conditioning

$$\mathbb{E}_1[B_0 B_t \mathbb{1}_{\{|\tilde{T}_t| = z\}}] = \mathbb{E}_1[\mathbb{1}_{\{|\tilde{T}_t| = z\}} \mathbb{E}_1[B_0 B_t \mid |\tilde{T}_t| = z]]. \tag{4.9}$$

Now given the size of the forward edge boundary $(z + 1)$, we know from the setup prior to this lemma that B_0 and B_t are two components of a multinomial distribution, therefore

$$\begin{aligned}\mathbb{E}_1[B_0 B_t \mid |\tilde{T}_t| = z] &= Cov(B_0, B_t \mid |\tilde{T}_t| = z) \\ &\quad + \mathbb{E}_1[B_0 \mid |\tilde{T}_t| = z]\mathbb{E}_1[B_t \mid |\tilde{T}_t| = z] \\ &= -(z+1)p_t^2 + (z+1)^2 p_t^2 \\ &= z(z+1)p_t^2. \qquad (4.10)\end{aligned}$$

As (4.10) is deterministic, we can substitute it into (4.9) to conclude the proof. $\square$

4.6 Proofs of other results required for Conjecture 4.1

In this section we prove the results that bridge the gap between Conjectures 4.8 and 4.1, as well as technical results needed so that the second moment method works as intended.

4.6.1 Proof of Lemma 4.9: moving from B_{m_ε} to $\tilde{B}^\varepsilon_{m_\varepsilon}$

We'd like to use the second moment method we've proved for progeny to therefore give us exceptional times regarding the position of the leftmost particle. Sadly the argument we will use revolves around forests surviving a certain number of generations, which is not the same, but they are indirectly related. Recall that for fixed $\varepsilon > 0$.

$$\begin{aligned}\tilde{A}^\varepsilon_j(t) &:= A_j(t) \cap \{j\text{th forest survives} \geq e^{(\gamma-\varepsilon)^j} \text{ generations}\} := A_j(t) \cap F^\varepsilon_j(t), \\ \tilde{B}^\varepsilon_k(t) &:= \bigcap_{j=0}^{k} \tilde{A}^\varepsilon_j(t).\end{aligned}$$

In order to show that

$$\limsup_{n\to\infty} \frac{M_n(t)}{\log\log n} \leq \frac{1}{\log\gamma}$$

it suffices to show that for all $\varepsilon > 0$

$$\limsup_{n\to\infty} \frac{M_n(t)}{\log\log n} \leq \frac{1}{\log(\gamma-\varepsilon)}$$

which corresponds to $F^\varepsilon_j(t)$ above. Recall that we need a lower bound for our first moment, and an upper bound for our second moment. Given Conjecture 4.8, we have a trivial upper bound for our second moment (trivial in the sense that we just forget about the event $F_j^\varepsilon(t)$). So it remains to work on the first moment, which is:

$$\mathbb{P}(\tilde{B}^\varepsilon_{m_\varepsilon}(0)) = \mathbb{P}(\tilde{A}^\varepsilon_0(0)) \prod_{j=0}^{m_\varepsilon-1} \mathbb{P}\left(\tilde{A}^\varepsilon_{j+1}(0) \,\middle|\, \bigcap_{i=0}^{j} \tilde{A}^\varepsilon_i(0)\right).$$

$\mathbb{P}(\tilde{A}^\varepsilon_0(0))$ has a constant lower bound just like $\mathbb{P}(A_0(0))$ did ($F_0(0)$ is simply the event the process does not instantly die). Focusing on a general term in the telescope, we have

$$\begin{aligned}
\mathbb{P}\left(\tilde{A}^\varepsilon_{j+1}(0) \,\middle|\, \bigcap_{i=0}^{j} \tilde{A}^\varepsilon_i(0)\right) &\geq \mathbb{P}\left(\tilde{A}^\varepsilon_{j+1}(0) \,\middle|\, \bigcap_{i=0}^{j} A_i(0)\right) \\
&= \mathbb{P}\left(A_{j+1}(0) \,\middle|\, \bigcap_{i=0}^{j} A_i(0)\right) \mathbb{P}\left(\tilde{F}^\varepsilon_{j+1}(0) \,\middle|\, \bigcap_{i=0}^{j+1} A_i(0)\right) \\
&\geq \mathbb{P}\left(A_{j+1}(0) \,\middle|\, \bigcap_{i=0}^{j} A_i(0)\right) \\
&\qquad \cdot \min_{z \in [e^{\gamma j}, 2(1+\epsilon_j)e^{\gamma j}]} \mathbb{P}_z(F^\varepsilon_{j+1}(0) \mid A_{j+1}(0)).
\end{aligned}$$

Recall that the $(j+1)$th dynasty depends on previous dynasties only through the size of the forward edge boundary of the jth dynasty, see Corollary 4.4. With this in mind, the first inequality comes from the fact that forcing the previous dynasty to survive a long time will reduce the number of trees (as each $A_i(0)$ restricts the total progeny) which in turn reduces the size of the forward edge boundary. This idea also provides the minimum in the second inequality. The first term in this final line is precisely what is in Lemma 4.12, so all we need to prove is that the product over j of the "min" term is a constant, and then the second moment method words with the exact same ordered bounds as it did previously. We look to simplify this probability, using Bayes' theorem on its complement, so we want to pick z that maximises an upper bound:

$$\begin{aligned}
\mathbb{P}_z(F^c_{j+1}(0) \mid A_{j+1}(0)) &= \frac{\mathbb{P}_z(A_{j+1}(0) \mid F^c_{j+1}(0))\mathbb{P}_z(F^c_{j+1}(0))}{\mathbb{P}_z(A_{j+1}(0))} \\
&\leq \frac{\mathbb{P}_z(E_{j+1}(0) \mid F^c_{j+1}(0))}{\mathbb{P}_z(A_{j+1}(0))}. \qquad (4.11)
\end{aligned}$$

Note we have suppressed the ε in the notation above for the sake of readability. The numerator of (4.11) is (where Z_i represents the ith generation of the critical Galton-

Watson process generating this all):

$$
\begin{aligned}
&\le \mathbb{P}_z(\text{prog} > e^{\gamma^{j+1}} \mid \text{survive} \le e^{(\gamma-\varepsilon)^{j+1}} \text{ gens}) \\
&\le \mathbb{P}_z(\exists i \le e^{(\gamma-\varepsilon)^{j+1}} \ : \ Z_i > e^{\gamma^{j+1}}/e^{(\gamma-\varepsilon)^{j+1}}) \\
&\le z\mathbb{P}_1(\exists i \le e^{(\gamma-\varepsilon)^{j+1}} \ : \ Z_i > e^{\gamma^{j+1}-(\gamma-\varepsilon)^{j+1}}) \\
&\le z \sum_{i=1}^{e^{(\gamma-\varepsilon)^{j+1}}} \mathbb{P}_1(Z_i > e^{\gamma^{j+1}-(\gamma-\varepsilon)^{j+1}}) \le z e^{2(\gamma-\varepsilon)^{j+1}-\gamma^{j+1}}
\end{aligned}
$$

where we've used the union bound in the penultimate line, and Markov's inequality in the final inequality (noting that a critical Galton-Watson process has mean generation size 1). Focusing on the denominator of (4.11) now, by (4.6) (noting $A''_{j+1}(0) \subset A_{j+1}(0)$)

$$
\mathbb{P}_z(A_{j+1}(0)) \ge \frac{2\varepsilon_{j+1}z}{\sqrt{\pi}} e^{-\frac{1}{2}\gamma^{j+1}}(1 + O(\varepsilon_{j+1})).
$$

In total, (4.11) reads

$$
\mathbb{P}_z(F^c_{j+1} \mid A_{j+1}) \le \frac{\sqrt{\pi}}{2\varepsilon_{j+1}(1 + O(\varepsilon_{j+1}))} e^{2(\gamma-\varepsilon)^{j+1}-\frac{1}{2}\gamma^{j+1}}
$$

which is independent of z. As this decays to zero very fast (as $\varepsilon_j = j^{-2}$ and γ^{j+1} is much larger than $(\gamma-\varepsilon)^{j+1}$) we have that

$$
\begin{aligned}
\prod_{j=0}^{m_\varepsilon - 1} \min_{z \in [e^{\gamma^j}, 2(1+\epsilon_j)e^{\gamma^j}]} \mathbb{P}_z(F^\varepsilon_{j+1} \mid A_{j+1}) &= \prod_{j=0}^{m_\varepsilon - 1} \min_{z \in [e^{\gamma^j}, 2(1+\epsilon_j)e^{\gamma^j}]} (1 - \mathbb{P}_z(F^c_{j+1} \mid A_{j+1})) \\
&\ge \prod_{j=0}^{m_\varepsilon - 1} \left(1 - \frac{\sqrt{\pi} e^{2(\gamma-\varepsilon)^{j+1}-\frac{1}{2}\gamma^{j+1}}}{2\varepsilon_{j+1}(1 + O(\varepsilon_{j+1}))} \right) \\
&= C > 0
\end{aligned}
$$

where the constant is due to the product being decreasing in n, but converges to a non-zero constant by some elementary analysis of expressions of the form $\lim_n \prod_j^n (1 + x_j)$. This proves that the second moment method works with $(\tilde{B}^\varepsilon_{m_\varepsilon(n)}(t))$, given Conjecture 4.8.

4.6.2 Proof of Lemmas 4.10 and 4.11

Proof of Lemma 4.10. We define

$$T_n := \{t \in [0,1] \ : \ M_j(t) \leq m_\varepsilon(j) \ \forall j \leq n\}$$

and we aim to show that

$$\bigcap_n \bar{T}_n = \bigcap_n T_n \ a.s.$$

We briefly recall the dynamical process. We have a binary tree and attached to each edge is a dynamical random variable $X_{i_1 i_2 \dots}(t)$ which takes values 0 and 1 with probability 1/2 each, independently of each other. Rerandomisations are done according to separate PPP(1)s. If we can show for each N that

$$\bigcap_{n \geq N} (\bar{T}_n \setminus T_n) = \emptyset \ a.s.$$

then as the sets (T_n) and $(\bar{T}_n)$ are nested we get that

$$\Big(\bigcap_{n \geq 1} \bar{T}_n\Big) \setminus \Big(\bigcap_{n \geq 1} T_n\Big) \subset \bigcup_{N \geq 1} \bigcap_{n \geq N} (\bar{T}_n \setminus T_n)$$

which is empty, giving us the conclusion we require.

We now proceed, as each edge rerandomises independently from each other, we have that a.s. only one rerandomisation can happen at any individual time. Equally the system only changes when a rerandomisation takes place, so

$$\bar{T}_n^\alpha \setminus T_n^\alpha \subset \{\tau_{i,j}^{(k)} : k = 1, 2, \dots, \ i = 1, \dots, n, \ j = 1, \dots, 2^i\}$$

where $\tau_{i,j}^{(k)}$ represents the time of kth rerandomisation of the jth edge (counting from left-to-right) connecting from $Gen(i-1)$ to $Gen(i)$. If τ is a time where a rerandomisation takes place, say on an edge from $Gen(k-1)$ to $Gen(k)$ ($k \geq 1$) connecting to a vertex $v \in Gen(k)$. The subtree with root v is the only subtree of nodes/particles effected by this one edge rerandomising, and the evolution of the edges in this subtree are independent of the rest of the edges.

Treating the subtree with root v as its own tree, we can use Bramson's result, Theorem 1.10, to say that for sufficiently large n, every particle from $Gen(n)$ onwards (relative to v) are at distance

$$d \geq \frac{\log\log(n)}{\log(2)}$$

from v (the left most particle itself from v turns the inequality to an equality). Now viewing this binary tree with respect to the entire tree. We can say that at time τ and for large n that every particle in $Gen(n)$ within the subtree containing v is at a distance

$$d \geq \frac{\log\log(n-k)}{\log(2)} + S_v(\tau)$$

from the true root. The $n-k$ corresponds to the two $Gen(n)$'s being with respect to different roots, and $S_v(\tau)$ is the random value contributed by edges prior to v. Now

$$S_v(\tau) = S_v(\tau^-) \pm 1$$

depending on whether we move from a 0 to a 1 or vice-versa. Irregardless, the left-most particle within the subtree containing v is at distance

$$d \geq \frac{\log\log(n-k)}{\log(2)} + S_v(\tau) \geq \frac{\log\log(n-k)}{\log(2)} - 1 > \frac{\log\log(n)}{\log(\gamma-\varepsilon)}$$

for all large n (k is fixed and $\gamma - \varepsilon > 2$). So that $\tau \notin \bar{T}_n \setminus T_n$ for large n, as required. $\square$

Moving on to the proof of Lemma 4.11. We can't use an ergodicity argument as in the previous two chapters due to the branching nature of the process. However we can produce a 0-1 law using a similar idea.

Proof of Lemma 4.11. Recall in our process that each generation each particle produces two offspring, so the total number of particles brought into existence at generation k is 2^k, or $|Gen(k)| = 2^k$. For $u \in Gen(k)$ and $\varepsilon > 0$, define

$$H_u^\varepsilon(t) := \left\{ \forall \text{large } n,\ u \text{ has a descendent } w \in Gen(n) \text{ s.t. } S_w(t) < \frac{\log\log(n)}{\log(\gamma-\varepsilon)} \right\}.$$

Denoting the root by 0, we know (given Conjecture 4.8) that for all $\gamma > 2$

$$\mathbb{P}(\exists t \in [0,1]\ :\ H_0^\varepsilon(t) \text{ occurs}) \geq C > 0.$$

We want to show that the above equals one, so it suffices to show that it satisfies a 0-1 law. It ends up boiling down to whether we can show that for all $u \in Gen(k)$ ($\forall k$) we have

$$\mathbb{P}(\exists t \in [0,1]\ :\ H_u^\varepsilon(t) \text{ occurs} \mid \mathcal{F}_k) \geq \delta > 0$$

where $\mathcal{F}_k$ means that we have all knowledge the the branching process up until and including the kth generation for all times t. It is vital that δ is independent of k. If

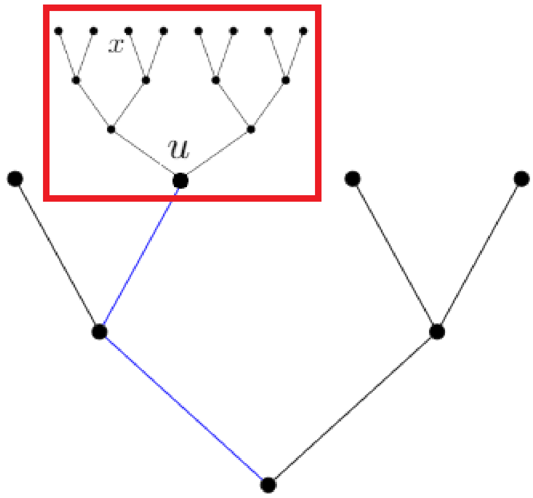

Figure 4-4: To obtain the value of $S_x(t)$, you need to add $S_u(t)$ (the blue edges) onto the edge values connecting u to x.

the above holds then by Levy's upward theorem (see Chapter 14.2 in [53]) we have that $(\mathbb{P}(\exists t \in [0,1] \ : \ H_u^{\varepsilon}(t) \text{ occurs} \mid \mathcal{F}_k))_k$ is a uniformly integrable martingale that is uniformly bounded away from zero and converges a.s. to

$$\mathbb{P}(\exists t \in [0,1] \ : \ H_u^{\varepsilon}(t) \text{ occurs} \mid \mathcal{F}_\infty) = \mathbb{1}_{\{\exists t \in [0,1] \ : \ H_u^{\varepsilon}(t) \text{ occurs}\}} = 1$$

where $\mathcal{F}_\infty$ is the smallest $\sigma-$algebra generated by $(\mathcal{F}_k)_k$. The above is valid as $\{\exists t \in [0,1] \ : \ H_u^{\varepsilon}(t) \text{ occurs}\}$ is $\mathcal{F}_\infty-$measurable and the martingale is uniformly bounded away from zero.

We now need to show that indeed

$$\mathbb{P}(\exists t \in [0,1] \ : \ H_u^{\varepsilon}(t) \text{ occurs} \mid \mathcal{F}_k) \geq \delta > 0.$$

What we do here is relate it to $H_0^{\varepsilon}(t)$ by treating u as the root of the subtree consisting of all descendants of u. The value (in the branching random walk) of a descendant $x \in Gen(n)$ of u has two components:

- The component of the walk prior to (and including) u, which is deterministic due to $\mathcal{F}_k$ and is given by $S_u(t)$.
- The component of the walk actually in the subtree with u as the root.

See Figure 4-4 for a visual representation of this. Viewing u as the root of its own tree and recalling the defintion of $m_\varepsilon(n)$, we can reformulate $H_u^\varepsilon(t)$ as:

$$H_u^\varepsilon(t) = \{\forall \text{large } n,\ u \text{ has a descendent } w \in Gen(n) \text{ s.t. } S_w(t) < m_\varepsilon(n+k) - S_u(t)\}$$

where the change of n to $n+k$ accounts for the fact that $Gen(n)$ now is $Gen(n)$ relative to u, and we also need to subtract the component of the walk prior to u. Now $0 \leq S_u(t) \leq k$ at all times (and k is fixed) so

$$H_u^\varepsilon(t) \supset \left\{\forall \text{large } n,\ u \text{ has a descendent } w \in Gen(n) \text{ s.t. } S_w(t) < \frac{\log\log(n+k)}{\log(\gamma-\varepsilon)} - k\right\}.$$

Now for all large n we know that

$$\frac{\log\log(n+k)}{\log(\gamma-\varepsilon)} - k \geq \frac{\log\log(n)}{\log(\gamma-\varepsilon/2)}$$

which can be seen by writing the difference between the left and right-hand sides as

$$\log\log(n)\left[\log(\gamma-\varepsilon/2)\frac{\log\log(n+k)}{\log\log(n)} - \log(\gamma-\varepsilon)\right] - k\log(\gamma-\varepsilon)\log(\gamma-\varepsilon/2)$$

noting that the term inside the square brackets is eventually positive, so the entire first term is growing in n, while the term being subtracted is fixed. Therefore $H_u^\varepsilon(t) \supset H_0^{\varepsilon/2}(t)$, so the probability that there exists a t where the former occurs is at least as large as the latter, which we know is a constant. $\square$

4.7 Conjecture 4.2: deviations of M_n

4.7.1 Motivation behind Conjecture 4.2

Conjecture 4.1, if true, states that almost surely exceptional times exist where $M_n(t) \leq \log\log(n)/\log(\gamma)$. A loose interpretation of this result is that at these exceptional times we require the mth dynasty to survive for $\exp(\gamma^m)$ generations. To achieve this, we can ask the jth dynasty to have progeny roughly equal to $\exp(2\gamma^j)$ for all $j \leq m$. This interpretation is not really what happens in our outline for a proof of Conjecture 4.1, since we introduced ε to give ourselves breathing space between progeny and survival events, but it is the vague idea behind the proof.

Conjecture 4.2 says something very different. Ignoring the dynamical model, if we simply condition the mth dynasty to survive for $\exp(\gamma^m)$ generations (therefore obtaining $M_n \leq m$) then we see vastly different behavior than described above. In particular, the jth dynasty has progeny roughly equal to $\exp(2\gamma^m 2^{j-m})$ for $j \leq m$. This configuration of progenies is never seen in our dynamical model, as for this to occur we would require the 0th dynasty to have progeny $\exp(2(\gamma/2)^m)$ which (as $m \to \infty$ as $n \to \infty$) corresponds to requiring the root of a critical binary tree to be in an infinite component of zero-valued edges. By [36, Theorem 1.3] no such exceptional times exist. To be more precise, the event providing the second moment argument for Conjecture 4.1, $\tilde{B}^{\varepsilon}_{m_\varepsilon}(t)$, satisfies

$$\mathbb{P}(\tilde{B}^{\varepsilon}_{m_\varepsilon}(0)) \text{ is roughly } \exp\left(-\frac{1}{2}\left(\frac{\gamma-2}{\gamma-1}\right)\gamma^m\right)$$

while Conjecture 4.2 states that

$$\mathbb{P}(M_n \leq m) \text{ is roughly } \exp\left(-\left(\frac{\gamma}{2}\right)^m\right).$$

The latter is much larger, we can therefore deduce that the dynamical model at exceptional times does not behave like the system conditioned to have its left most particle leftwards of $m(n, \gamma)$. Instead, it takes a far less likely configuration.

Before proving the more interesting direction of Conjecture 4.2 in the next subsection, we briefly emphasise the different ways that our successive dynasties can grow. To simplify our heuristic, we state everything in terms of progeny rather than surviving a certain number of generations, noting that for a critical Galton-Watson process, the progeny is roughly the square of how long the process survives. This is indicated by Yaglom's limit law [33, Theorem C], which states that

$$\frac{1}{n}Z_n|\{Z_n > 0\} \to Exp(2\sigma^{-2})$$

in law, suggesting that the progeny grows like n^2. We also use a simplified version of Corollary 4.4, and say that the number of initial particles of the $(j+1)$th dynasty is equal to $\mathcal{P}^{(j)}$, the total progeny of the jth dynasty.

Firstly we take the case where we do not condition $M_n \leq m$, so the dynasties grow naturally. If the 0th dynasty has constant progeny P, then the progeny of the 1st dynasty (which starts with P particles) is roughly P^2. For the 2nd dynasty we have progeny roughly equal to P^4, and so on. This can be seen as $z = \sqrt{a}$ maximises Lemma 4.5, so it is most probable for successive progenies to square in size if there is no conditioning taking place.

We move on to the strategy in Conjecture 4.1, which is that the jth dynasty has progeny roughly $\exp(\gamma^j)$. The 0th dynasty has a constant progeny P, and then with the above formula we see that the jth dynasty must have progeny roughly P^{γ^j}. As $\gamma > 2$, we see that each dynasty in this strategy is a bit larger than what happens in the unconditional case. That is, at exceptional times the burden of growth is given somewhat equally to each dynasty.

Finally, we stated that for the growth as in Conjecture 4.2 we need the jth dynasty to have progeny roughly equal to

$$P^{(j)} := \exp\left(\frac{2\gamma^m}{2^{m-j}}\right).$$

Notice that the progeny of the $(j+1)$th dynasty is still roughly the square of the jth dynasty. The difference is that the 0th dynasty needs progeny $\exp(2(\gamma/2)^m))$ which as $\gamma > 2$ is very large. This means that when conditioning on $M_n \leq m$, we put all the burden of growth on the 0th dynasty, and let all other dynasties grow naturally.

The moral of the story is therefore that the most likely way a static event may occur is not necessarily the way that exceptional times in a dynamical model may form. The dynamical model prefers to spread the burden of growth among all dynasties, rather than focusing in on the 0th dynasty.

4.7.2 Proof of the lower bound of Conjecture 4.2

Recall that we can view our BRW model as a series of Galton-Watson forests (called dynasties) $F^{(0)}, F^{(1)}, \dots$ positioned at $0, 1, \dots$. Each tree within a dynasty is generated using IID copies of the critical Galton-Watson branching process with offspring distribution with generating function

$$g(s) = \frac{1}{4} + \frac{1}{2}s + \frac{1}{4}s^2.$$

The initial population of each dynasty satisfies $Z_0^{(0)} = 1$ for $F^{(0)}$ and for $i \geq 1$ we have that $F^{(i)}$ has $Z_0^{(i)} = \mathcal{P}^{(i-1)} + Z_0^{(i-1)}$, where $\mathcal{P}^{(j)}$ denotes the total progeny of the dynasty $F^{(j)}$. Recall that the object we care about is M_n the position of the left-most particle in our BRW after n steps. A sufficient condition for $\{M_n \leq m\}$ to hold is that the forest $F^{(m)}$ survives at least n generations. Consider the event:

$$\left\{\mathcal{P}^{(k)} \geq \exp\left(\frac{2\gamma^m}{2^{m-k}}\right)\right\}.$$

As discussed in the previous subsection, if this event occurs when $k = m$ then so should $\{M_n \leq m\}$. We now state and prove the lower bound:

Theorem 4.18. *Let $\gamma > 2$ and $m(n, \gamma) = \frac{1}{\log \gamma} \log \log n$ (written m for legibility). Then* *we have* $\forall n \geq 1$ *that*

$$\mathbb{P}(M_n \leq m) \geq \frac{2}{\sqrt{\pi}} C^m \exp\left(-\left(\frac{\gamma}{2}\right)^m\right)\left(1 + O\left(e^{-\frac{2\gamma^m}{2^m}}\right)\right)$$

where $C > 0$ *is a finite constant.*

Proof. The idea is that for a critical Galton-Watson process, surviving n generations is similar to having total progeny n^2, and the collection of events

$$\bigcap_{k=0}^{m-1} \left\{\mathcal{P}^{(k)} \geq \exp\left(\frac{2\gamma^m}{2^{m-k}}\right)\right\}$$

for the dynasties up to $m-1$ naturally lead to the mth dynasty having progeny n^2. We compute

$$\begin{aligned}
\mathbb{P}(M_n \leq n) &\geq \mathbb{P}(Z_n^{(m)} > 0) \\
&\geq \mathbb{P}\left(\{Z_n^{(m)} > 0\} \cap \bigcap_{k=0}^{m-1} \left\{\mathcal{P}^{(k)} \geq \exp\left(\frac{2\gamma^m}{2^{m-k}}\right)\right\}\right) \\
&= \mathbb{P}\left(Z_n^{(m)} > 0 \,\middle|\, \bigcap_{k=0}^{m-1} \left\{\mathcal{P}^{(k)} \geq \exp\left(\frac{2\gamma^m}{2^{m-k}}\right)\right\}\right) \\
&\qquad \cdot \mathbb{P}\left(\bigcap_{k=0}^{m-1} \left\{\mathcal{P}^{(k)} \geq \exp\left(\frac{2\gamma^m}{2^{m-k}}\right)\right\}\right) \\
&\geq \mathbb{P}_n(Z_n^{(m)} > 0)\mathbb{P}\left(\bigcap_{k=0}^{m-1} \left\{\mathcal{P}^{(k)} \geq \exp\left(\frac{2\gamma^m}{2^{m-k}}\right)\right\}\right) \qquad (4.12)
\end{aligned}$$

where the first term in the last line comes from the fact that to minimise $\mathbb{P}(Z_n^{(m)} > 0)$, we should start off with as few particles as possible, and as the conditioning states that $\mathcal{P}^{(m-1)} \geq \exp(\gamma^m) = n$, we must start with at least n particles (in fact we start with a lot more). Now Lemma 4.6 gives that the first term is at least $1/2$. By telescoping we get that the second term is equal to

$$\mathbb{P}_1\left(\mathcal{P}^{(0)} \geq \exp\left(\frac{2\gamma^m}{2^m}\right)\right) \prod_{k=1}^{m-1} \mathbb{P}\left(\mathcal{P}^{(k)} \geq \exp\left(\frac{2\gamma^m}{2^{m-k}}\right) \,\middle|\, \prod_{j=0}^{k-1} \left\{\mathcal{P}^{(j)} \geq \exp\left(\frac{2\gamma^m}{2^{m-j}}\right)\right\}\right)$$

Again, there is monotonicity we can exploit, in the sense that if $Z_0 > \exp\left(\frac{2\gamma^m}{2^{m-k+1}}\right)$ then the $Z_0 - \exp\left(\frac{2\gamma^m}{2^{m-k+1}}\right)$ extra starting individuals will contribute a positive number of individuals to the total progeny. By the same reasoning as earlier, the minimum number of starting particles that can contribute towards $\mathcal{P}^{(k)}$ given the conditioning is $\exp(2\gamma^m 2^{-m+k-1})$, therefore:

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{P}^{(k)} \geq \exp\left(\frac{2\gamma^m}{2^{m-k}}\right) \middle| \prod_{j=0}^{k-1}\left\{\mathcal{P}^{(j)} \geq \exp\left(\frac{2\gamma^m}{2^{m-j}}\right)\right\}\right) \\
&\geq \mathbb{P}_{\exp\left(\frac{2\gamma^m}{2^{m-k+1}}\right)}\left(\mathcal{P} \geq \exp\left(\frac{2\gamma^m}{2^{m-k}}\right)\right).
\end{aligned}$$

Writing $r = \exp(2\gamma^m 2^{k-1-m})$ for convenience, let A be a large finite constant such that for $a \geq Az^2$ the $O(z^2 a^{-1})$ error term appearing in Lemma 4.5 has absolute value at most 1/2. Then Lemma 4.5 gives

$$\begin{aligned}
\mathbb{P}_{\exp\left(\frac{2\gamma^m}{2^{m-k+1}}\right)}\left(\mathcal{P} \geq \exp\left(\frac{2\gamma^m}{2^{m-k}}\right)\right) &\geq \mathbb{P}_r(\mathcal{P} \geq Ar^2) \\
&\geq \frac{2(r+1)}{\sqrt{Ar^2\pi}}(1 + O(A^{-1})) \\
&= \frac{2(1+r^{-1})}{\sqrt{A\pi}}(1 + O(A^{-1})) \\
&\geq \frac{1+r^{-1}}{\sqrt{A\pi}} \\
&\geq \frac{1}{\sqrt{A\pi}} =: C > 0.
\end{aligned}$$

We can also use Lemma 4.5 on the 0th dynasty (noting it starts with 1 particle) to get

$$\mathbb{P}\left(\mathcal{P}^{(0)} \geq \exp\left(\frac{2\gamma^m}{2^m}\right)\right) \geq \frac{4}{\sqrt{\pi}} \exp\left(-\left(\frac{\gamma}{2}\right)^m\right)\left(1 + O\left(e^{-\frac{2\gamma^m}{2^m}}\right)\right).$$

Inserting all of these facts into (4.12) gives

$$\mathbb{P}(M_n \leq m) \geq \frac{2}{\sqrt{\pi}} C^m \exp\left(-\left(\frac{\gamma}{2}\right)^m\right)\left(1 + O\left(e^{-\frac{2\gamma^m}{2^m}}\right)\right)$$

as required. □

Chapter 5

Conclusion And Open Questions

In this thesis we have studied three different models related to the fields of noise sensitivity and exceptional times in deep detail. The primary motivation for this thesis was to study whether two objects with the same distribution (as an entire sequence) could exhibit differing levels of sensitivity to noise, or equivalently, when allowed to evolve dynamically. This led us to the compass and switch random walks (Y_n) and (Z_n), the former studied in [7] by Benjamini et al, and the latter in Chapter 2 of this thesis as well as in [40] by Roberts and Prigent. We have seen that (Z_n) is noticeably more sensitive to noise, and also exhibits exceptional times where (Y_n) does not.

A natural extension to the work above is to consider Brownian motion, as it is the scaling limit of a random walk. Indeed, in Chapter 3 we proved results for Brownian motion that are similar to the ones in Chapter 2. This involved reworking a few definitions (mainly regarding the influence) and setting up a two-dimensional Poisson point process for the dynamics. It is worth noting that our construction has the potential to be used with other stochastic processes.

Finally, we expanded our reach to a model more complex than a simple symmetric random walk. We took a branching random walk within the class studied by Bramson [11] and aimed to study its dynamics thoroughly. While our main result regarding exceptional times is only a conjecture, we have dealt with all elements of the proof bar the upper bound on the second moment, and have analysed the distribution of the 0th dynasty at both times 0 and t simultaneously. We expect this to be vital tool for proving Conjecture 4.1.

We now briefly highlight potential avenues of interest for further development.

5.1 Questions related to Chapter 2

5.1.1 Extensions of Theorem 2.1

A question that arises immediately from Theorem 2.1 is whether or not there exist exceptional times where

$$\liminf_{n\to\infty} \frac{Z_n(t)}{n^\alpha} > 0$$

for the critical value $\alpha = 1/2$. α above or below this critical value is dealt with in the theorem itself. We are not sure whether or not this holds, as we would need to examine the sample paths of the random walk in far more detail than we currently do.

In a similar spirit to the above, a corollary of Theorem 2.1 is that the law of the iterated logarithm (LIL) is a dynamically sensitive property for the switch random walk. To better understand what happens to the random walk at these exceptional times, it is worth investigating the value of

$$\inf_{t\geq 0} \limsup_{n\to\infty} \frac{Z_n(t)}{\sqrt{2n\log\log n}}.$$

Due to Theorem 2.1, we know that there almost surely exists a time t where $Z_n(t) < 0$ for all $n \geq 1$, so the infimum above must be non-positive. Is this infimum zero? On a similar note, does there exist some function f (independent of t) such that

$$\inf_{t\geq 0} \limsup_{n\to\infty} \frac{Z_n(t)}{f(n)} = -1?$$

5.1.2 The p-switch walk

As we have seen, the compass and switch walks (Y_n) and (Z_n) act quite differently when noised despite having the same law. There is a way of interpolating between these two walks, so that we can study how "close" to (Z_n) the noise sensitivity begins.

We do this by defining $X_1, X_2, \dots$ as IID Rademacher random variables, and make them dynamical in the standard way. We fix $p \in [0,1]$ and define $\tilde{X}_i^{(p)}(t)$ as follows. For $i = 1$ it is simply $X_1(t)$ while for $i \geq 2$ we have

$$\tilde{X}_i^{(p)}(t) = \begin{cases} X_i(t) & \text{with probability } 1-p \\ \tilde{X}_{i-1}^{(p)}(t)X_i(t) & \text{with probability } p \end{cases}$$

We can then define $Z^{(p)} = (Z_n^{(p)}(t))_{n,t}$ to be the dynamical simple symmetric random walk where $Z_0^{(p)}(t) = 0$ and

$$Z_n^{(p)}(t) = \sum_{i=1}^{n} \tilde{X}_i^{(p)}(t)$$

Note that when $p = 1$ we have the walk $(Z_n(t))$ while when $p = 0$ we have $(Y_n(t))$, so what we have defined is a natural way of interpolating between the dynamical compass and switch walks. We have the following conjecture:

Conjecture 5.1. *The events* $(\{Z_n^{(p)} > 0\}, n \geq 1)$ *are maximally noise sensitive for* $p = 1$*. That is, for any sequence* $(\varepsilon_n)_{n\geq 1}$ *in* $(0, 1)$ *such that* $n\varepsilon_n \to \infty$*, we have that as* $n \to \infty$

$$\mathbb{P}(Z_n^{(1)}(0) > 0 \textit{ and } Z_n^{(1)}(\varepsilon_n) > 0) - \mathbb{P}(Z_n^{(1)}(0) > 0)^2 \to 0.$$

For $p \in [0, 1)$ *these events are noise stable.*

We believe this conjecture to be true, which would mean that there is an extremely sharp phase transition between noise stability and *maximal* noise sensitivity.

The reason we believe this conjecture holds is since a change from X_i to $-X_i$ in this model causes a reflection of length of law $Geom(1 - p)$, which is finite for $p < 1$. This means that the noised portions of the walk should be small in length (for small noise parameter ε) and thus contribute little to the value of $Z_n^{(p)}$. The main technicality preventing a proof from being immediate is that in the definition of noise stability, the supremum is taken over n before ε is taken to zero.

5.1.3 Biased switch random walks

In the spirit of [15] we can change the law of our Rademacher steps to generate different switch random walks. If $X_1, X_2, \ldots$ to be IID and biased such that

$$\mathbb{P}(X_1 = 1) = p = 1 - \mathbb{P}(X_1 = -1)$$

then (Z_n) given by $Z_n = \sum_{i=1}^{n} \prod_{j=1}^{i} X_j$ has a distribution that is a function of p. Note that the walk studied in Chapter 2 is with $p = 1/2$. It may be of interest to discover for what values of p there exist exceptional times of recurrence/transience. Before we do that, we must understand how the value of p affects recurrence in the static model.

Note that when $p \neq 1/2$ the walk (Z_n) is not a Markov process, but the pair $(Z_n, Z_{n+1})_n$ is, see [15, Remark 1.3]. Therefore we now explicitly define what we mean by recurrence, which is that

$$\mathbb{P}(Z_n = 0 \ i.o.) = 1$$

Returning to our question of when (Z_n) is recurrent, it is actually an established result that goes back to at least the works of Gillis [20]. This is because our biased switch random walk is in fact a case of the Gillis-Domb-Fisher correlated random walk on $\mathbb{Z}$. We summarise the result from Gillis, in our case, as follows:

Proposition 5.2. *$(Z_n)_{n\geq 0}$ is recurrent for $p < 1$ and transient for $p = 1$.*

For the interested reader, recurrence of our process and more general processes can be viewed through the lens of a class of additive models studied in [43] by Rogers.

Correlated random walks more generally are sometimes referred to in the literature as "persistent" or "Newtonian" random walks. A useful general reference on these walks is [41]. Similar continuum models are also studied in the physics literature and are known as "run-and-tumble processes".

When considering whether or not exceptional times exists, one important thing to note is that the bias in the Rademacher steps is also present in the rerandomisation, i.e. when a Poisson clock rings, that Rademacher step becomes $+1$ with probability p, not $1/2$. Whether this prevents exceptional times from occurring or not is unclear, except for the cases $p = 0, 1$ where rerandomisations do not change the process. In [15] the authors use a sequence (p_n) of numbers in $[0, 1]$ to define $\mathbb{P}(X_n = 1)$ for each n. We could also ask similar questions for switch random walks defined by such a sequence, and check for dynamical sensitivity of the results in [15].

5.1.4 A short note on the two dimensional switch random walk

In order to justify our construction in the next section of the two dimensional dynamical Brownian motion, it is worth briefly discussing the two dimensional switch random walk. We can define a switch random walk on $\mathbb{Z}^2$ as follows; Let $X_1, X_2, \dots$ be independently, identically distributed random variables taking each of $e_1, -e_1, e_2, -e_2$ with probability $1/4$, where e_i is the ith standard basis vector. Now define

$$Z_n = \sum_{i=1}^{n} \prod_{j=1}^{i} X_j$$

where the multiplication is defined by the rules:

$$e_1^2 = -e_2 \quad e_1^3 = -e_1 \quad e_1^4 = e_2$$

i.e. the multiplication is the cyclic group of order 4, $\langle e_1 \rangle$, with identity e_2. e_1 can be thought of as rotating your current position by 90 degrees clockwise, and taking a step. This means that $-e_2$ corresponds to turning around and taking a step back, $-e_1$ means turn 90 degrees anti-clockwise and take a step, while e_2 simply means to take a step in the direction you're already travelling in. Note that at the start of the process you are facing in the e_2 direction. We can make the walk dynamic via

$$Z_n(t) = \sum_{i=1}^{n} \prod_{j=1}^{i} X_j(t)$$

where the Rademacher's rerandomise in the usual manner. A change via rerandomisation in this model leads to the path from the point of the change being rotated by a number of degrees equal to the angle between the old and new vector, a multiple of 90 degrees.

The reason we have not studied this process in this thesis is because the two dimensional compass walk is already maximally dynamically sensitive, in that the set of exceptional times of transience has Hausdorff dimension 1 [23, Theorem 1]. So, as the switch walk is more sensitive than the compass walk, we expect the following:

Conjecture 5.3. *The two dimensional switch random walk almost surely has exceptional times of transience. The set of such times almost surely has Hausdorff dimension one.*

5.2 Questions related to Chapter 3

Firstly, as with the switch random walk, a question to which we cannot conjecture the answer is whether or not there exist times where

$$\liminf_{s \to \infty} \frac{B_s(t)}{\sqrt{s}} > 0.$$

One extension to this model we could consider is looking at two dimensional dynamical Brownian motion. This would be constructed by taking a realisation of a two dimensional Brownian motion as the dynamical process at time zero, and then equipping it with a three dimensional Poisson point process on $[0, \infty) \times [0, \infty) \times [0, 2\pi)$. The first two dimensions correspond to where and when changes occur (i.e. Brownian and dynamical time). The third dimension corresponds to the angle at which the future path is rotated by, stated clockwise with respect to the standard axes in $\mathbb{R}^2$. It may be possible to model all orthogonal transformations in $\mathbb{R}^2$ (combinations of rotations and reflections) on the Brownian path, but we stick to solely rotations in the above description as we do not have reflections in the two dimensional switch walk presented in the previous subsection.

The motivation behind studying the two dimensional dynamical Brownian motion is that the distinction between recurrence and transience is more nuanced for Brownian motion. In particular, [35, Theorem 3.19] states that two dimensional Brownian motion is neighbourhood recurrent but not point recurrent. This means that such a process almost surely hits every open ball infinitely often, but only hits individual points only finitely often. We have the following conjecture

Conjecture 5.4. *There almost surely exists exceptional times where the dynamical two dimensional Brownian motion is not neighbourhood recurrent. The set of such times almost surely has Hausdorff dimension one.*

We believe this conjecture holds as in the proof of [23, Theorem 1], for the two dimensional compass random walk, it is shown that there are times where the walk diverges without returning to zero. Noting that switch dynamics are more extreme than compass dynamics, and converting from random walks to Brownian motion is what motivates this conjecture.

5.3 Questions related to Chapter 4

Obviously the top priority is to ascertain whether or not Conjectures 4.1 and 4.2 truly hold. Matthew Roberts and myself hope to prove this in the affirmative in the near future. Given these results, it is still unknown what speed $s(n)$ is the phase transition for exceptionality. By this we mean that it is desirable to find a function $s(n)$ such that there are no exceptional times where M_n diverges slower than $s(n)$, but there are almost surely exceptional times where M_n diverges faster than $s(n)$, but still slower than the standard $(1/\log 2)\log\log n$ speed. It is clear given Conjecture 4.1 that $\log\log(n)/s(n) \to \infty$. We also know that $s(n) \to \infty$, i.e. there are not exceptional times where M_n does not diverge. This is because if $\lim_n M_n < \infty$ then there must be a time where the binary tree has an infinite component of zero-valued edges, which cannot occur by [36, Theorem 1.3]. While we do not believe that there are exceptional times where M_n diverges faster than normal, it would be interesting to see if that is indeed true. Also we have not asked any noise-sensitivity questions of this process yet.

It would also be interesting to extend our dynamical model to the entire class of branching random walks considered by Bramson. As a reminder, the static general model can be viewed as a Galton-Watson tree with offspring distribution L, conditioned on non-extinction, where each edge is attached with an independent random variable with law X, where for some $p \in (0,1)$:

$$\mathbb{P}(X = 0) = p = 1 - \mathbb{P}(X > 0).$$

Bramson's result, Theorem 1.10, then holds provided $p\mathbb{E}[L] = 1$. The largest difference between this setup and the model in Conjecture 4.1 is that the binary tree we work with would instead be a random tree conditioned on non-extinction. A good starting point would be to fix the random tree at time 0 and have solely the edges evolve over time, as well as making X a binary variable e.g. taking only 0 and 1 as values. However, it should be possible to loosen these requirements.

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